Vagueness in mathematics talk

Thesis

UNRESTRICTED

Timothy Rowland, B.Sc., M.Sc.

VAGUENESS IN MATHEMATICS TALK

Thesis submitted for the degree of

Doctor of Philosophy

in

Mathematics Education

Author number: M 9328911
Date of submission: 17 November 1995
Date of award: 10 May 1996

November 1995

Dedicated to my parents, Muriel and Reg,

who believed it was achievable; and to

my wife, Judy, who made it possible.

CONTENTS

Fuzzy set theory 58

Hedges 59

LIST OF FIGURES,TABLES AND PLATES

ABSTRACT

'Vagueness in Mathematics Talk'

The Cockcroft Report claimed that "mathematics provides a means of communication which is powerful, concise and unambiguous". Such precision in language may be a conventional aim of mathematics, particularly when communicated in writing. Nonetheless, as this thesis demonstrates, vagueness is commonplace when people *talk* about mathematics.

In this thesis, I examine the circumstances in which vagueness arises in mathematics talk, and consider the practical purposes which speakers achieve by means of vague utterances in this context. The empirical database, which is considered in Chapters 4 to 7, consists almost entirely of transcripts of mathematical conversations between adult interviewers (including myself) and one or two children. The data were collected from clinical interviews focused on a small number of tasks, and from fragments of teaching. For the most part, the pupils involved in the study were aged between 9 and 12, although the age-range in Chapter 7 extends from 4 to 25.

I draw on a number of approaches to discourse associated with 'pragmatics' - a field of linguistics - to analyse the motives and communicative effectiveness of speakers who deploy vagueness in mathematics talk. I claim that, for these speakers, vagueness fulfils a number of purposes, especially 'shielding', i.e. self-protection against accusation of being wrong. Another purpose is to give approximate information; sometimes to achieve shielding, but also to provide the level of detail that is deemed to be appropriate in a given situation. A different purpose, associated with a particular form of vagueness (of reference), is to compensate for lexical gaps in pursuit of effective communication of concepts and ideas. I show, in particular, how speakers use the pronouns 'it' and 'you' in mathematics talk to communicate concepts and generalisations.

Some consideration is given to the intentions of 'expert' speakers of mathematics when they deploy vague language. Their purposes include some of those identified for novices. Teachers also use vagueness as a means of indirectness in addressing pupils; this strategy is associated with the redress of 'face threatening acts'.

My thesis is that vagueness can be viewed and presented, not as a disabling feature of language, but as a subtle and versatile device which speakers can and do deploy to make mathematical assertions with as much precision, accuracy or as much confidence as they judge is warranted by both the content and the circumstances of their utterances.

I report on the validation and generalisation of my findings by an Informal Research Group of school teachers, who transcribed and analysed their own classroom interactions using the methods I had developed.

ACKNOWLEGEMENTS

A great many people have enabled and assisted my work for this thesis. No matter how I try, I am a debtor unable to repay what I have been given. I accept and acknowledge all these gifts with gratitude.

To my wife, Judy who agreed that I should take on this project, even though she knew what it would cost her; who arranged access to the children in her classes and those of her colleagues; who is the first reader and critic of almost everything that I write; who has understood and supported my commitment to this work.

To our sons Mark and Simon. To Simon, for being 'Simon' when I began, and my desktop publishing consultant *par excellence* when I was finishing. To Mark for his sensitive contribution to Chapter 6, and for his constant interest in my work.

To my supervisor, Dr David Pimm, for saying "yes" when I asked him to supervise me; who guided, encouraged and corrected me throughout, whilst always sustaining my self-belief; for being wise whilst carrying wisdom so lightly; for being, at all times, the ultimate academic editor. I claim sole responsibility for any surviving flaws of form, content or taste.

To my external supervisor in linguistics, Dr Margaret Deuchar, for accepting the daunting invitation to guide a mathematician in the ways of language; who enabled me to make up for lost time where pragmatics is concerned; for constant courtesy and encouragement.

To Eric Love, for some months my 'locum' supervisor, whose talent for noticing crucially directed the early course of my research.

To Homerton College, for funding both the study reported in Chapter 6, and also my 'writing up' study leave in the Easter Term of 1995.

To many colleagues at Homerton College and co-researchers at the Centre for Mathematics Education, Open University; especially for listening, and for much good and expert advice.

To Rosemary Hay, headteacher, the staff and the pupils of St Luke's Primary School, Cambridge; for generous access to classrooms, and many wonderful mathematical conversations with children.

To the schoolteachers who became members of the Informal Research Group - especially Judith Addley, Kevin Gault, Hazel Matthews, Ann Neale, Sue Ray and Rachel Williams; for affimation through participation.

To my parents, who first taught me to take both pleasure and care in the use of words.

PAGE
MISSING
IN
ORIGINAL

TRANSCRIPT CONVENTIONS

Whilst a good deal of transcribed speech is used in evidence in this thesis, I have kept the conventions used in such transcripts to a minimum. One advantage of this decision is that the text can be read more fluently. A disadvantage is the loss of many *nuances* of speech, particularly of intonation.

Conventions:

S3	the coded name of the transcript to which the speech belongs - all such transcripts are listed in Appendix 1.
12	number of the speaker's turn in the transcript.
S3:12	the 12th turn in transcript S3.
John:	the name or code-name of the speaker.
[points]	description of non-linguistic communication.
[inaudible]	indecipherable speech
[*her teacher*]	transcriber's situational elucidation or comment.
[...]	transcript ellipsis of words or turns.
so ... we	'...' indicates a short (untimed) pause or hesitation.
[pause]	a longer pause or silence.
so ...	utterance interrupted and/or not completed.
/ /	encloses utterance overlapping that of next or previous speaker.
think	(italic) word or words stressed by the speaker.
think	(bold) highlighted by the transcriber for the attention of the reader.

Then said the teacher, 'Speak to us of Teaching'. And he said:

> 'No man can reveal to you aught but that which already lies half asleep in the dawning of your knowledge.
>
> The teacher who walks in the shadow of the temple, among his followers, gives not of his wisdom but rather of his faith and his lovingness.
>
> If he is indeed wise he does not bid you enter the house of his wisdom, but rather leads you to the threshold of your own mind'.

[Kahlil Gibran, 1926: *The Prophet*]

I study mathematics as a product of the human mind and not as absolute.

[Emil Post]

Pilate said to him, "What is truth?"

[John 18:38]

Man: What is your disappointment in life? Your major one, I mean, if I may ask?

Woman: [...] Oh, I don't know. Not getting more than three O levels, I suppose.

Man: Which?

Woman: Sorry?

Man: Which three?

Woman: English language, English literature. And French.

Man: Good. Good. Language orientated.

[Dennis Potter, 1991: *Secret Friends*]

INTRODUCTORY PREFACE

In 1981, I wrote an article (Rowland, 1982) about the teaching of directed numbers: what they 'are' and how to add and subtract them. Essentially it proposed a 'concrete' model of the integers. The learner was intended to become acquainted with the model with a view to performing 'natural' actions on it for the purposes of extending arithmetic on natural numbers. I reported and evaluated an empirical study of an implementation of the model.

From 1986 to 1990, the SCDC/NCC-funded curriculum development project *Primary Initiatives in Mathematics Education* (PrIME) was based at Homerton College, Cambridge, where I work. One major component of PrIME was the introduction of a calculator-aware number curriculum (CAN) in selected schools. In 1986, I began a study of children aged six and seven in three CAN schools. My report was completed in 1988 and reprinted as Rowland (1994b).

The CAN children were using the usual icons and models for natural numbers (principally number lines and Dienes blocks), but they also had free access to calculators. Amongst other things, I was struck by the variety of their personal mental methods, and by the sensible and cogent use made of calculators by all but a few of the children. This contrasted starkly with their reluctance to use the manipulatives ('apparatus') with which they were supposedly familiar, even when specifically urged by their teachers to use them. I commented:

> If children are to learn through "experience", we have to ask what kinds of experience, and not to overlook the importance of internal mental activity, possibly prompted and supported by a calculator, in the construction of mathematical knowledge. (p. 25)

I subsequently reformulated this account:

> The calculator consistently demonstrates mathematical structures and rules about how numbers are represented and how they behave. However, *it does not impose on the user any specific form of concrete imagery with which to think about numbers*. Thus, children are given maximum intellectual freedom to set each new experience with a calculator alongside their other experiences, and thus to construct, modify and add to the frameworks of meaning they already possess (Rowland, 1990, pp. 1-2)

Articles by Pimm (1986), Fielker (1988) and others resonated with my observations and my developing thinking in this area. Statements such as:

> maybe all that children learn from manipulating blocks is an understanding of how to manipulate blocks. (Fielker, 1988, p. 6)
>
> Far from mathematics being all around us, I offer the alternative tenet that mathematics is only inside us. (Pimm. 1986, p. 51)

seemed to capture what I was coming to believe, whilst at the same time appearing to undermine some conventional tenets of primary mathematics practice.

One consequence of these converging influences was personal rejection of the tacit epistemological and pedagogical assumptions which underlay my 1982 article - namely, that pupils learn by accepting the meanings (of mathematical objects, operations, and so on) that teachers provide for them, clean and pre-processed. In effect, my earlier belief was that it is the teacher's job to make sense of mathematics on behalf of the pupil, and that the didactical transposition (Chevallard, 1985) of mathematical knowledge is a matter of neat, even ingenious, packaging and careful training.

Certainly I had managed to train some children to exhibit some of the behaviours I had wanted in relation to the addition and subtraction of integers, but now I wondered what meanings those children attributed to the mechanics of solving the 'problems' I posed them. The CAN experience in particular convinced me that children may well eventually reject the teacher's pre-selected, pre-packaged mathematics in preference for their own personal and private meanings, constructed from the *totality* of their experience.

My aim and desire, by about 1988, was to access and describe the mathematical frameworks and private constructions locked away in children's minds; for, as far as teaching is concerned, this is the Aladdin's cave.

> Everyone - almost everyone - agrees that our goal is the study of the mathematical mind in action. (Ginsburg, 1981, p. 4)

At the outset of the period of research which culminated in this thesis, my approach was guided by two fundamental principles:

Linguistic principle: language is one important means of access to thought.

Thus, by talking with children, I would (with suitable reflection) be given insight into the structure of fragments of their mathematical understanding. The roots of the principle are in Freudian psycho-analysis and its provenance as a research method in education goes back at least to Piaget and famously bears fruit in mathematics education in Ginsburg (1977). The work of Douglas Farnham (1975) belongs to a strand of work with an explicitly linguistic foundation. Farnham drew on contemporary work of Barnes, Coulthard and others on patterns of classroom interaction to account for the child's development of mathematical understanding in terms of social sense-making.

> **Imagery principle:** mathematical thought about entities and relationships is structured and accomplished by reference to personal, internal images of various kinds.

Thus, imagery is the key to describing mathematical understanding and performance. For examples of studies based on this principle, see Plunkett (1979), Ernest (1983) or Presmeg (1986).

In 1991, I spent a term working in a school with a class of eight- and nine-year-old children. I began systematically to tape record and transcribe my teaching sessions and one-to-one interviews with these children. Two things emerged. On the one hand, I achieved little success in eliciting substantial descriptions of any relevant imagery from the children. On the other, I experienced a growing awareness (which I describe first in Chapter 4) that the language the children used when talking about mathematics, whilst revealing little about imagery, was of considerable interest in its own right - not in the sense that I had originally expected (by providing descriptions of images), but in the subtle ways that these children *used* language to point to private concepts, meanings, beliefs, feelings or attitudes in the context of their mathematical thinking.

Consequently, I held on to the linguistic principle (and will discuss it in detail) as the foundation stone of my research, but was forced to abandon (but not to deny) the imagery principle, replacing it with:

> **Deictic principle:** speakers use language for the explicit communication of thought, and as a code to express and point to concepts, meanings and attitudes.

I coined the name because it derives from the Greek word *deiknumi*, meaning 'to show' or 'to point', and is associated with a linguistic tem, 'deixis', which features in Chapter 4. The centrality of the linguistic and deictic principles to my research orientation will be examined further in the next chapter in a discussion of the clinical interview, and again in Chapter 3, where I shall set out some linguistic interpretive tools.

A NOTE ON 'PRAGMATIC' AND RELATED TERMS

In this thesis, variants of the word 'pragmatic' are used in three different, but related, senses. The first two are technical terms.

Pragmatism (or Pragmaticism) is the name of a philosophical position due to the American polymath Charles Sanders Peirce (1839-1914). It is intended to be a 'practical' (as opposed to 'theoretical') kind of philosophy, which embeds rational discourse in life and conduct. The name 'pragmatism' derives from the Kantian term *pragmatisch*, expressing relation to some human purpose. The essence of pragmatism is that human rationality and purpose are inseparable. Peirce's ideas have evolved through William James, John Dewey and others, and are central to the philosophical foundations of the interpretivist research paradigm (Giarelli, 1988).

Pragmatics is the name of a branch of linguistics which attempts to interpret the meaning of utterances by reference to the motives of speakers and to context of use. The distinction between syntax, semantics and pragmatics goes back ultimately to Peirce's theory of signs, or semiotics (Lyons, 1977, p. 114). Indeed, Morris described pragmatics as "the relation of signs to interpreters" (1938, p. 6), although these Peircean origins are now more or less irrelevant (Lyons, 1977, p. 119). The domain of pragmatics can usefully be viewed as those aspects of meaning that cannot be dealt with by means of truth-conditional semantics (Gazdar, 1979, p. 2).

The adjective corresponding to both the philosophical position *and* the branch of linguistics is '**pragmatic**'. I am also obliged to use the same word in the everyday, non-technical sense of real-world realism, utilitarian, sometimes in contrast to 'ideal'.

I intend that the particular use will be clear from the context; the first two meanings will be recapitulated and expanded when the context requires in subsequent chapters.

CHAPTER 0: PREVIEW AND METHODOLOGY

> It [APU practical testing] afforded an opportunity to hold a prolonged mathematical conversation with a child. My understanding of children's thought processes when solving problems has been considerably extended. (A teacher, quoted in Foxman *et al.*, 1980, p. 73).

INTRODUCTION

In the preface, I described the aim which motivated the early stages of this research, namely to access and describe the mathematical frameworks and private constructions locked away in children's minds. My concern at that time was to uncover what they 'knew'. and how they structured that knowledge. I saw this as the most likely kind of outcome (in the spirit of the linguistic principle) of the mathematical conversations in which I planned to engage them. In other words, I began with my attention focused on that function of language that Brown and Yule (1983, pp. 1-4) call 'transactional':

> That function which language serves in the expression of content we describe as *transactional*, and that function involved in expressing social relations we will describe as *interactional*.
>
> Whereas linguists, philosophers of language and psycholinguists have, in general, paid attention to the use of language for the transmission of 'factual propositional information', sociologists and psycholinguists have been particularly concerned with the use of language to negotiate role-relationships, peer-solidarity, the exchange of turns in a conversation, the saving of face of both speaker and hearer.

The two categories of language function are not exclusive, and I shall argue that both are of the utmost importance in talk about mathematics. But my initial interest in transactional elements is reflected in my implementation of variants of Piaget's clinical interview, which I discuss later in this chapter, in many conversations with children. This research orientation is most strongly represented in Chapter 4.

Thereafter, my analytical focus shifts towards interactional components of mathematics talk. In particular, I began to explore (in the spirit of the deictic principle) how speakers in such conversations show their concern for a number of pragmatic goals, principally those to do with the saving of 'face' (Goffman, 1967).

The deictic principle is at the heart of a paper in which the linguist Michael Stubbs (1986) draws attention to a number of ways, many of them relevant to this thesis, in which speakers use language to convey beliefs and attitudes, or to distance themselves from the propositions they make. (A new presentation of the paper will appear as Chapter 8 of Stubbs, 1996.) This epistemic subtext is sometimes summed up in the phrase 'propositional attitude' (Ginsburg *et al.*, 1983, p. 26) which is glossed, in effect, by Sperber and Wilson (1986a, pp. 10-11) as follows:

> Utterances are used not only to convey thoughts but to reveal the speaker's attitude to, or relation to, the thought expressed; in other words, they express 'propositional attitudes' [...]

Stubbs asserts that no utterance is neutral with regard to the belief and commitment of the speaker, and urges the importance of the study of markers of propositional attitude:

> whenever speakers (or writers) say anything, they encode their point of view towards it: whether they think it a reasonable thing to say, or might be found to be obvious, questionable, tentative, provisional, controversial, contradictory, irrelevant, impolite, or whatever. [...] All sentences encode such a point of view [...] and the description of the markers of such points of view and their meanings should therefore be a central topic for linguistics. (1986. p. 1)

Stubbs identifies *vagueness* and *indirect* language as a principal means of encoding propositional attitude. In fact, vagueness is the linguistic feature which unifies the data which I analyse in this thesis; more precisely, it is vague aspects of the language of participants in mathematical conversation that I shall single out for my analytical attention. My principal reason for choosing that particular focus is that I came to see the significance, for mathematics talk, of Stubbs' insight about the encoding of propositional attitude. More surprisingly, I came to perceive how vagueness, suitably deployed, can also assist the transactional purposes of mathematics talk. The main and subordinate aims of my thesis are best understood in the light of these surprising perceptions.

AIMS AND THEMES OF THE THESIS

My overall aim is to expose and understand some of the ways that participants in mathematics talk use language - especially vague language - to achieve their communicative and affective purposes. This comprehensive aim guides my choice of subject matter, and finds empirical expression in work that I shall present as four studies, reported in Chapters 4 to 7. Each study was motivated by a particular sub-aim, related to the main one.

First, in this Chapter 0 and in Chapter 3, I review the methodological and linguistic matters which underpin the design and interpretation of each of the four studies. Given the unifying theme of this thesis, I discuss some mathematical and philosophical dimensions of vagueness in Chapter 2. The mathematical process of generalisation features strongly in three of the four studies; since I hold the view that this process encapsulates the essence of mathematical thought, I have devoted Chapter 1 to an exploration and exposition of its unique character.

Chapter 4 is principally a detailed study of one nine-year-old child, Susie. My aim In that chapter is to demonstrate the transactional effectivenes of the pronouns 'it' and 'you' in our mathematical conversations. The vagueness of these words is associated with reference indeterminacy. I shall show how the first of these pronouns enables Susie to introduce certain concepts and generalisations into our conversation, despite the fact that she has no name for them. I shall demonstrate that the second is associated with vagueness-as-generality, and that 'you' surfaces in children's mathematics talk as a natural language pointer to generalisation.

The study in Chapter 5 is based on several similar conversations with pairs of children aged 9 to 11. The similarity lies in in the fact that each begins from the same numerical-combinatorial task, designed to provoke generalisation. A paper of George Lakoff (1972) had first alerted me to a linguistic feature of the transcripts of these conversation, namely 'hedges' (such as 'I think', 'maybe', 'about' and 'around'). My aim in this study is to identify the use and prevalence of hedges in connection with conjectures. I shall show, as Stubbs suggests, that such hedges are powerful indicators of propositional attitude. In particular they point to vulnerability, they protect against loss of face. I shall introduce a construct which I call the Zone of Conjectural Neutrality, a space in which conjectures can be tested whilst minimising the affective risk to their originators.

In Chapter 6, I report a large-scale study which aims to trace the development, between the ages 4 and 11, of modal language competence, especially in the use of modal auxiliaries and hedges. The mathematical activities entailed in this study are counting and estimation. I shall identify a trend towards a developing ability to indicate propositional attitude in these ways in the primary years, and an increasing awareness that vagueness is essential to estimation. I shall identify an anomaly in this trend, and account for it by reference to institutional factors.

The final empirical study, in Chapter 7, examines a disparate collection of teaching episodes across a wide age-range, and involving eight different teachers. Here, the aim is validate three claims which arise from the findings in the earlier chapters. First, it is a demonstration of the applicability of the linguistic toolkit which I have assembled, to the analysis of transactional and interactional features of transcripts of talk in mathematical interaction. Second, it confirms the prevalence and interactional significance of a number of previously-identified (in the earlier studies) aspects of vague and indirect language in mathematics talk, across a wide age-range. Third, by involving a group of teachers to work with me for that study, I was able to validate the relevance of my methods and findings for their day-to-day work in the classroom. The justification of this claim of relevance is implicit in Chapter 7 and explicit in the final chapter.

A CONTEXT IN LANGUAGE RESEARCH

The linguist Joanna Channell has researched aspects of vague language for some fifteen years. Channell concludes her recent book (1994, p. 209) with a call for more research in 'variation study', including "the study of occurrence of vagueness in different registers or genres". The term 'register' refers to the specialised language peculiar to certain user-groups (Halliday *et al.*, 1964). Such studies of aspects of vague spoken language exist in certain academic fields, for example medicine (Prince *et al.*, 1982) and biology (Dubois, 1987). One aspect of variation not specifically included by Channell in her call for research is age-variation: the informants in every empirical study of vagueness of which I am aware are adults. I believe that this thesis makes a contribution to the study of vague language in these two dimensions of variation - register (mathematical) and age (especially five- to eleven-year-olds).

This work is a thesis in mathematics education, not linguistics. But mathematics education is necessarily and beneficially an eclectic discipline; the relevance of linguistics to this thesis is principally interpretive, in that certain organising principles of language use, particularly those which have become associated with the young linguistic science of 'pragmatics' (Levinson, 1983; Mey, 1993), are applied to make sense of some vague features of mathematics talk identified in various corpora.

Hymes (1972) has claimed:

> Studying language in the classroom is not really 'applied' linguistics; it is really basic research. Progress in understanding language in the classroom is progress in linguistic theory. (p. xviii)

As it happens, most of the language data for my study were gathered in the course of 'interviews' in classrooms, rather than in naturalistic classroom settings *per se*. I see this study as basic research, not in linguistic theory, but in mathematics education. A number of regularities of speech discovered in the data will be described and interpreted as phenomena observable in the interaction between individuals as they talk about mathematics. Some evidence of the presence and broader consideration of the pedagogic significance of these phenomena in the interaction between practising teachers and pupils is offered in Chapter 7. The thesis concludes with some proposals for the application of this basic research in the cause of improving the teaching of mathematics, informed by my commitment to a constructivist view of learning and a quasi-empiricist philosophy (this philosophy is described in Chapter 2) of the learning of mathematics. In fact, there is already evidence (reported in Chapters 7 and 8) that teachers other than myself might be able to 'use' what I have learned from this research in the classrooms where they teach.

The ability of speakers to encode ideas, commitments, attitudes and beliefs in their utterances - not least, in teachers' and pupils' mathematical utterances - would be of little use if their interlocutors were unable to decode, interpret and understand the intended subtext in such utterances. The communication and interpretation of propositional attitude is central to mathematics education because the articulation of beliefs, conjectures and even 'answers' in mathematics is notoriously a risk-taking activity; this point is developed further in Chapter 5. Acknowledging that some relevant groundwork has been done in linguistics, David Pimm identifies interpretation, within classroom discourse, as an area which is now ripe for research effort:

> I predict the extremely subtle pragmatic interpretive judgements regularly made by both teachers and pupils in the course of mathematics teaching and learning in classrooms will move steadily to the fore as a research topic. (Pimm, 1994, p. 167)

In summary, this thesis could be viewed as

- a *response* to Stubbs' call for the description of markers of propositional attitude, in the specific context of mathematics;
- a *contribution* to variational study (Channell) of vague spoken language - in the domain 'mathematics'; *and*
- a partial *fulfilment* of Pimm's prediction concerning research in mathematics education.

MATHEMATICAL CONVERSATIONS

For the most part, the data examined in this thesis consist of transcripts of mathematics talk. Much of this mathematics talk took place in the context of clinical interviews. In this chapter, I shall discuss the application of this interview method to research in mathematics education, and describe some principles which underpin my interpretation of the transcript data.

The 'linguistic principle' which I asserted in the preface reflects my confidence in the value of talking to pupils and students with a view to accessing their mathematical thinking. It is an conviction widely shared by researchers into children's thinking, deriving from a sense of the benefit of personal communication with the subject:

> It is my belief that the researcher can best formulate and test hypotheses and interpret the results of the tests in intensive interactive communication with the child, so that a close personal and trusting relationship can be formed. (Steffe, 1991, p. 178)

An approach to the study of children's thinking through 'interviews' with them is closely associated with Jean Piaget.

Piagetian Legacy

In 1920, at the age of 23, Piaget moved from post-doctoral study of philosophy of science and pathological psychology at the Sorbonne to a post at the Binet Laboratory in Paris (Piaget, 1952a, p. 244). [Note 0.1] His task was to standardise Cyril Burt's reasoning tests on Parisian children. In the administration of such tests, the wording and format of the questions were precisely defined, and had to be adhered to by the tester to safeguard the reliability of the procedure. Differences in performance between children (scaled by various "measures") were calculated by reference to the correct responses given by them. Piaget, however, found the *incorrect* answers much more interesting, since they caused him to wonder what kind of reasoning gave rise to them. He realised, moreover, that such questions could not be researched by means of standardised tests.

> The first method that presents itself is that of tests [... in which] the question and the conditions in which it is submitted remain the same for each child [...] But for our particular purpose the test method has two important defects [...] the essential failure of the test method in the researches with which we are concerned is that it falsifies the natural mental inclination of the subject [...] The only way to avoid such difficulties is to vary the questions, to make counter-suggestions, in short, *to give up the idea of a fixed questionnaire.* (Piaget, 1929, pp. 3-4, my emphasis)

He concludes, having considered and dismissed both the 'test' method and 'pure observation', that a third approach may be deployed, one which exploits the best of each of the rejected methods whilst avoiding their disadvantages.

> This is the method of clinical examination, used by psychiatrists as a means of diagnosis [... in which] the good practitioner lets himself be led, though always in control, and takes account of the whole of the mental context. Since the clinical method has rendered such important service in a domain where formerly all was disorder and confusion, child psychology would make a great mistake to neglect it. (*ibid.*, pp. 7-8)

Piaget's interest in psychiatry originated in his mother's mental illness which significantly coloured his own childhood (Piaget, 1952a, p. 238), but in fact the (then) novel clinical methods of psycho-analysis were under wide discussion as to their educational application, and not only in Europe (Mackie, 1923).

Contingent Questioning

Piaget's introduction to *The Child's Conception of the World* (1929) contains his only discussion of the clinical interview as a research methodology. The method was subsequently developed and adapted (or 'revised' - see Ginsburg *et al.*, 1983, p. 10) by Piaget from pure adult-child discourse to include manipulation of materials so that actions as well as words are added to the interpretive data bank. Piaget's classical work using the revised method is that on conservation e.g. Piaget and Szeminska, 1952. The clinical method became the basis of Piaget's work for half a century. Ginsburg (1981) and others (Ginsburg *et al.*, 1983) argue strongly for the efficacy of the method in research into children's mathematical thinking

Ginsburg is perhaps best-known in this field for his classic 1977 book *Children's Arithmetic,* in the preface to which he is explicit:

> The primary method is the in-depth interview with children as they are in the process of grappling with various sorts of of problems [...] Interviews like these, involving close observation of individuals, are rare in mathematics education, but essential to improving it. (p. iv)

The clinical method is appropriate for the purposes of identifying (eliciting), describing and accounting for cognitive processes (Ginsburg *et al.*, 1983, pp. 11-13). In this thesis, such processes will include prediction, generalisation and explanation in mathematics. The description of 'intensive interviewing' in social science research, as given in Brenner (1985), has much in common with Ginsburg's account of the clinical method.

The characteristic dimensions of the verbal clinical interview [Note 0.2] are:

- the interviewer employs a *task or tasks* to channel the subject's activity: typically, the interviewer presents a problem of some kind at the outset of the interview. Subsequent tasks or problems will depend on the subject's reaction to the initial task.

- The interviewer's questions are *contingent* on the child's responses: indeed, after the initial task has been presented, the interviewer's whole contribution to the interaction is judged and decided on the basis of the subject's contribution. That is not to say that the interviewer necessarily surrenders control of the interview (see my comments on 'frame' later in this chapter), but that s/he constantly makes instantaneous decisions about her/his questions and the direction of the interview.
- There may be some degree of *standardisation*: the actual extent of standardisation will depend on whether the interview is intended to discover or to elucidate cognitive phenomena. For example, concerning behaviour which has been previously identified and considered, standardisation may assist 'explication' of behaviour or detailed study of the prevalence of some phenomena.
- The procedure demands *reflection*: the interviewer asks the subject to reflect on what s/he has done and to articulate her/his thoughts, typically by means of questions such as "How did you do that?", "Can you explain that?", and so on.
- The interviewer makes appropriate use of scientific *experimental method*, such as holding some variables constant whilst deliberately varying others. The contingent nature of the interviewer's responses enable her/him to test hypotheses that s/he has generated (either in the interview or as a consequence of reflection on previous interviews with the same subject) to account for cognitive processes or other phenomena which have been identified in this interview, or in earlier interviews.

(based on Ginsburg *et al.*, 1983, pp. 18-20)

Ginsburg insists that "contingency of questioning" (1981, p. 6) is at the heart of the method; that the essential and distinguishing feature of the clinical interview mode is the contingent (responsive, interactive) nature of the interviewer's contribution.

The contingent interviewer is like a barrister in court, having continually to make rapid assessments of what 'witnesses' say, to probe without leading the witness. Unlike the advocate, of course, the role of the clinical interviewer is not supposed to be adversarial (winning a case) but analytical, striving to create the conditions for the surface manifestation - especially in speech - of the subject's thought.

The Use of the Clinical Method in Testing

A form of contingent questioning which is well-known in the British context is the so-called Practical Testing mode used by the Mathematics Monitoring Team of the Assessment of Performance Unit (APU) in the decade 1978-1987. [Note 0.3] Alongside large-scale pencil-and-paper tests, the APU team developed a number of semi-structured, individual interviews based on practical tasks with weights, shapes, money, and so on. The purposes of these one-to-one interviews, which were unique among *national* assessment programmes, include "exploration of children's reasoning and understanding of mathematical ideas" (Foxman *et al.*, 1981, p. 4), an outcome which began to be stressed early in the APU testing programme. Despite the 'practical' label, the essential feature of this testing mode is that it is interactive. Having assigned a prepared task for the pupil, the tester (a teacher, trained for the role of 'practical tester') notes the child's responses. S/he may then offer 'prompts', to enable the pupil to reconsider an unprofitable strategy or to progress from a 'stuck' situation, and 'probes' to elicit or clarify the rationale underlying the pupil's response. For further discussion of the contingent questioning dimension of APU practical testing, see Rowland (1996).

"Long Practice" - Researching versus Teaching

The professional skills of teachers related to questioning potentially equip them particularly well to deploy this method, either for cognitive research or for diagnostic purposes. Indeed, the Department of Education and Science invested in the mid-1980s in the distribution of APU Practical Testing 'kits' to schools with this diagnostic purpose in mind. A clinical interview with a child may well result in learning *for* the child; the child may even perceive such an interviewer as a kind of 'teacher'. The primary purpose of the interview is, however, *to inform the interviewer* about the child. The cultural obstacle for teachers is the improbable notion of a sustained mathematical discussion with a child which is not (by intention) in some way an improving experience for the child. As Lynn Joffe, a member of the APU Mathematics Monitoring Team, observed, there is a powerful temptation for teachers to teach:

> Although it is extremely difficult, and is asking a lot, testers are asked, as far as they possibly can, [...] to suspend their inclination to teach [...] One way of getting round the urge to teach, is to try and substitute non-directive questions where one might be tempted to teach. (Joffe, undated, p. 3)

Earlier, Piaget had noted a related, but different, problem for teachers:

> the clinical method can only be learned by long practice [...] It is so hard not

> talk too much when questioning a child, especially for a pedagogue! (Piaget, 1929, pp. 8-9)

Piaget's remarks on the need for 'practice' convey the notion of the clinical interview (and in turn, by implication, the mathematical conversation) as a kind of art form, in which the artist (the interviewer) strives over time to develop and improve her/his performance. The analogy with questioning as a style of teaching is clear; an improvised, unique, oral 'performance' for which there are guidelines but no script. The development of the artistry through the study of tapes and transcripts is part of the satisfaction, for teaching as well as for researching. In the final chapter of her remarkable book *Wally's Stories*, an American kindergarten school teacher, Vivian Gussin Paley, explains how, in her classroom,

> the tape recorder preserves everything. It has become for me an essential tool for capturing the sudden insight, the misunderstood concept, the puzzling juxtaposition of words and ideas. I began to tape years ago [...] and I was continually surprised by what I was missing in all discussions. I now maintain a running dialogue with each tape as I transcribe its contents [...] *The tape recorder trains the teacher not the child*, who never listens to the tapes and who is curious about the machine only the first time. (Paley, 1981, pp. 217-8, my emphasis)

In Paley's book, episodes which are explicitly mathematical are the exception rather than the rule. She demonstrates, however, that - even in the routines of daily classroom events - talk, tape and transcripts can be a powerful means of researching and refining practice and of coming to understand children's thinking.

Conversations can be preserved as data, for later scrutiny, in the form of videotapes, audiotapes, field notes or transcripts (electronic or hard copy). Each of these media has advantages and disadvantages. The videotape, for example, includes non-verbal data (such as gestures and actions on materials) and seems to facilitate subsequent group consideration of features such as critical moments in the discourse; the audiotape preserves speech features such as intonation, pauses, and voice tone; the transcript, a transformation of the primary record of the event, focuses attention on the spoken word, or coded speech features. The transcript as electronic text file is invaluable to the computational linguist with an interest in (say) the relative frequency of use of certain words or grammatical structures. Since I have chosen to focus on spoken language, I principally audiotaped my data, and transcribed the tapes using a word processor.

CONTRARY TRENDS: FRAMING AND SAMPLE SIZE

Contingency and standardisation are contrasting and, inevitably, competing dimensions of the clinical method. Both are related to, but not in direct causal relationship with, the notion of control. Bernstein has offered a theoretical construct which he calls 'frame' to capture the essence of the control of knowledge in the teacher-pupil relationship.

> This frame refers to the degree of control teacher and pupil possess over the selection, organisation and pacing of the knowledge transmitted and received in the pedagogical relationship. (Bernstein, 1971a, p. 50)

Frame is a form of boundary, in a given context, between what is to be included and what is to be excluded. [Note 0.4] In the context of teaching and learning, where framing is 'strong', the fence around that which is to be learned is (supposedly) sharp, well-defined. Where framing is 'weak', the boundary is blurred, fuzzy. Thus, for example, 'investigational learning' in mathematics would seem to require a weakly-framed pedagogical relationship since, outside a core (possibly but improbably empty) of intended content learning outcomes, it is expected and hoped that pupil activity will result in the acquisition of other kinds of mathematical and strategic knowledge.

As I have already observed, research interviewing is not, by design, teaching, but I find it helpful to borrow the terms 'weakly-framed' and 'strongly-framed' to identify poles in a continuum of control exerted by the interviewer over the content and direction of the interview. The standard 'method of tests' (Piaget, 1929, p. 3), in which the interviewer's questions are scripted and the subject's responses possibly coded, must lie at or close to one pole (strongly-framed), in that the interviewer retains total control over the agenda. (Strictly speaking, the test designer has control *in absentia*, whilst the interviewer totally lacks control, since s/he has no discretion to deviate from the script.)

The empirical account in this thesis begins with just two children (Susie and Simon) and a sequence of extended one-to-one contingent interviews with each child. In no way are these interviews standardised; in each interview, only the initial task or question was pre-planned, and not one of the tasks was presented to both children. The interviews were weakly-framed in the sense that, beyond the initiation gambit, I had no pre-set agenda of my own for these interviews, no prepared schedule of questions or tasks, since the aim was the *discovery* of phenomena and related intellectual processes (Ginsburg, 1981, p. 5). It could be said, therefore, that Susie and Simon had a significant share in the determination and control of the agenda for these interviews. A major outcome (for me, as researcher) was the identification of particular linguistic pointers (surface phenomena) to generalisation (private mathematical process). These pointers are the subject of Chapter 4.

The next empirical stage in the research (Chapter 5) was designed to study the prevalence of such linguistic phenomena in relation to generalisation and associated mathematical processes. The sample size was increased to 20 children, who were interviewed in pairs to facilitate peer interaction. Given the sharper enquiry focus of the interviews, a standardised interview agenda was planned in the expectation that each conversation would proceed in 'phases' leading to conjectures and, in some cases, attempts at the explanation of 'rules'. Contingency remained an important factor of my role as interviewer, allowing in particular for differences in interpretation of the initial task. Nonetheless, in comparison with one-to-one interviews in the first stage, the framing was stronger. The class of linguistic pointers which were identified for study at this stage are called 'hedges' in the linguistics literature.

The final empirical stage - rather, the final empirical stage at which I exercised *any* control over the agenda for the interviews - was designed to test a hypothesis concerning children's use of hedges in the context of the mathematical process of numerical estimation. This study is reported in Chapter 6. The design at this stage required a much larger sample, in the event the whole population (230 children) of one primary school. Each interview needed to be relatively brief, typically five to ten minutes, with the questions standardised and focused on three prescribed tasks. A small measure of contingency was necessary, depending on each child's initial responses to each of the three tasks, with corresponding prompts requiring the child to reflect on her/his responses. Otherwise, little deviation from the tasks was permitted. Thus, the framing of these interviews was relatively strong, but not as strong as a standardised 'test' interview. [Note 0.5]

Two trends are therefore inherent in the design and administration of these clinical interviews. Whilst sample size increases from 2 to 230, the interviewer's control over the agenda and the data he (in this case) gathers - corresponding to Bernstein's notion of 'frame' - shifts from relatively weak to relatively strong. In other words, as the sample opens up, contingency closes down. A wish to display these two contrary trends and to preserve the chronological order of events between 1991 and 1995 has influenced my decision to present the three studies in Chapters 4, 5 and 6 in the order that I have.

ETHNOGRAPHY AND INTERPRETATION

Having accounted for the use of the verbal form of the clinical method as my principal means of data collection, I shall sketch here a broader framework of ideas which underpin my methodological commitment throughout this thesis. At the heart of this commitment is a belief that human events have no absolute 'meaning' [Note 0.6], but that it is possible that they be made meaning*ful* (connected to other agreed meanings) both individually and socially. That is to say that meaning is dependent on interpretation, which in turn is shaped by the world-view of the interpreter.

> I maintain that "critical reasoning" is an oxymoron, because consistent critical thinking shows that we are always inside our own vocabularies and our own angle on the world. We should give up the idea that we can somehow jump right out of our own limitations and achieve absolute knowledge, while yet remaining ourselves. (Cupitt, 1994, p. 20)

This is the perspective of interpretivism, a philosophical position often contrasted with logical positivism. Of course, the maintenance of personal sanity and inter-personal communication requires that social groups with a common interest - teachers, for example - normally go about their business as though consensual, interpreted meanings were absolute. This is the nature of inter-subjective knowledge. One of the signs of insanity and extremist politics is the inability to acknowledge the fallibility of interpreted reality.

Research into education, and mathematics education in particular, is necessarily an anthropological endeavour, since it entails the study of the behaviour of members of *homo sapiens* by members of the same species. The advantages and disadvantages of this peculiar state of affairs are evident. Quasi-scientific methods of research, deriving from experimental psychology, with arms-length collection of measurements from questionnaires and the like, may assist the researcher in achieving 'objectivity' through emotional detachment from the fellow creatures whom s/he is studying. Yet this in itself may be insufficient for the researcher to gain critical insight into the phenomenon that s/he has identified for study (Fischbein, 1990, p. 11).

Qualitative methods, which have grown enormously in use and acceptance over the last twenty years, are characteristically descriptive, inductive, speculative, interpretivist; drawing on 'naturalistic' data such as recordings made in working classrooms, case studies, extended but loosely-structured interviews, and from participant observation. Such methods permit the researcher to exploit his or her membership of and association with the species or sub-species (e.g. 'teacher') which s/he is studying; to get close to, to make contact with the context of study, or even to participate in that context.

Participation and Detachment

The research reported in this thesis sprang from cognitive ambitions. I never lost touch with those ambitions, nor did I lose my desire for insight into the mathematical mind in action. But within a year (by 1991), the focus of my work had shifted from the cognitive in the direction of the affective; and from preoccupation with the individual as an isolated thinker towards an interest in the interaction between individuals (more often than not, 'teachers' and 'pupils' of various kinds) when they talk about and do mathematics. As will become apparent, I make reference to linguistic forms in order to understand pedagogic interactions. This is *interpretive* in that the effort is oriented towards meaning-making, the goal is "knowledge about social action within a context" (Kilpatrick, 1988, p. 98). The action that I study is speech, focusing on aspects of language that can, for one reason or another, be classified as 'vague'; the context is people talking about mathematics.

Margaret Eisenhart (1988) gives a thorough survey of the interpretivist view of research and discusses some implications for research in mathematics education. Perhaps the most fully-committed form of interpretive research is in the ethnographic tradition, which emphasises a holistic understanding of some social group, an understanding achieved through immersion in and, to some extent, identification with, that social group. Eisenhart's article derives from an ethnographic perspective, but nonetheless confirms the conviction with which I began this research: that I would gain insight by interaction with the pupils, students and (as it turned out) the teachers whom I was studying.

> The purpose of doing interpretivist research, then, is to provide information that will allow the investigator to "make sense" of the world from the perspective of participants; that is, the researcher must learn how to behave appropriately in that world and how to make that world understandable to others, especially in the research community. (Eisenhart, 1988, p. 103)

I also recognised that the holistic goal of interpretivist research would best be served by both affirming and drawing on, rather than denying, my personal and professional identification with the enterprise which I was studying - the teaching and learning of mathematics. Such familiarity necessitates a determined effort of detachment by the researcher at certain points in the research [Note 0.7], otherwise s/he is unable to discern anything at all remarkable about the events s/he observes, even at a phenomenological level. Paul Atkinson (1981), discussing the study of classroom language, writes that:

> The very familiarity of mundane, ordinary social activity can be a great barrier to analysis. [...] One has to work rather hard to make the effort of will and imagination to render what is familiar *strange*. One has to approach the data as if one were an anthropologist, confronted with a new, alien and exotic culture, and hence suspend one's own commonsense, culturally given assumptions. This is what ethnomethodologists mean by the task of making everyday life 'anthropologically strange'. (p. 100)

A tape recording is a permanent, transformed record of an ephemeral event which had a sound (here, principally speech) component. Atkinson speaks of the making of transcripts of classroom talk as a 'discipline' (p. 99), in that it forces attention to details of the talk such as hesitation, interruption, false starts and incoherence. The listener's brain is inclined to 'tidy up' such details as a gratuitous, sense-making kindness when talk is experienced as a purely auditory event. If I had been present (usually as a participant) at the recorded event, a memory of the original context and the event remains. In the transcript, the word is made manifest as a random-access witness to the event. The text is now a transformed object in the world, tangible and *accessible for study in its own right.* For me, this transformation of tape into text is a significant means of achieving detachment from interactions in which I participated.

A Spiral Process

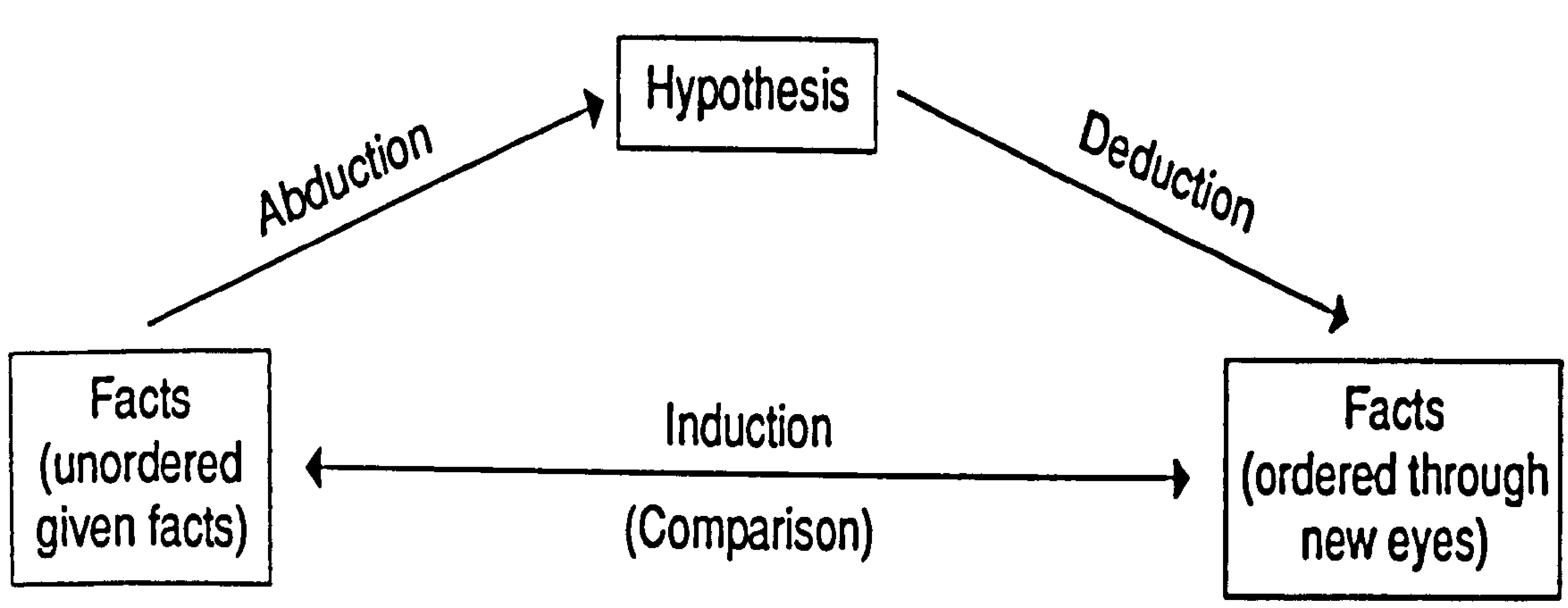
The interpretive challenge is to translate knowledge of the text into knowledge about the participants in the original context. Professional linguists have given little attention to mathematical discourse, perhaps because mathematics does not, on the whole, present itself as much of a social phenomenon as compared with, say, doctor-patient interviews. But because conversation about mathematics *is* nevertheless a social phenomenon, it would be expected to have some linguistic features of human interaction in common with some other social situations.

It would, I imagine, have been possible to identify (from the pragmatics literature, say) some linguistics concepts - person deixis and conversational implicature, for example - and to have sifted my data for examples of these concepts. This might have served the cause of pragmatics but probably not that of mathematics education. Instead, the approach I have taken is substantially inductive, drawing on a precept of 'grounded theory' (Glaser and Strauss, 1967) which recognises and legitimises the use of data as the *source* of research questions and hypotheses, not just a means of seeking answers to *a priori* questions and testing *a priori* hypotheses. For me, the transcript data is the source of my observations about linguistic phenomena in the text of the conversations. I may not know the 'technical' linguistic name of a phenomenon when first I notice it, or even know that it has a name. My interpretation of how and why these speakers use that feature of language in mathematical conversation, as an aspect of their communicative competence, is subsequently *informed by* my reading of related literature in pragmatics, as well as by the the usual diverse range of knowledge (mathematics, mathematics education, general education, psychology, philosophy, sociology, and so on) which comprises the broad research-cultural base of mathematics education.

In an absolute sense, of course, every hypothesis about the world arises from experience and in that sense from some kind of data. This observation is at the heart of the discussion of generalisation in Chapter 1. In the context of educational research, Rene Saran (1985) presents some remarks on grounded theory methodology which forge particular philosophical links between this chapter and the next two, making reference to:

> three methods which I use repeatedly in a spiral-like research process - abduction, deduction and induction. (p. 228) [Note 0.8]

The term 'abduction' is due to Peirce (1934, pp. 99 ff.) and refers to the process of hypothesis *formation*; the human mind 'invents' and proposes meaning (expressed as a hypothesis) as an imaginative leap from the data. The researcher applies deduction on returning to the data "*but with new eyes*, to [...] order the facts in a new way" (Saran, 1985, p. 229), whilst needing to be aware of the dangers of "an enchanting love affair with the hypothesis". For example, the creation of a classification of selected features of the data (abduction) could have the effect of relating previously unconnected elements of the data. Peirce reserves 'induction' to refer to hypothesis *testing* - making comparisons between the data, as perceived "with new eyes", and the previously unordered data, or indeed with additional data which was not used for abduction. Saran summarises the Peircean trichotomy in the following diagram (p. 230):



Linguistic theory plays a part in the process of abduction in this thesis. Awareness of regularities and theories of 'ordinary' language caused me to surmise that similar regularities might be found in mathematics talk and be accounted for in similar ways, but with a specifically mathematical dimension. Where this is the case, spoken mathematical interaction is indeed viewed with new eyes. The detailed application of the interpretive process is discussed further throughout the empirical sections of the thesis, especially Chapters 4, 5 and 6.

Hans Freudenthal has urged the possibility of viewing methodology as arising from *a posteriori* reflection on research activity.

> I don't remember when it happened but I do remember, as though it were yesterday, the bewilderment that struck me when I first heard that the training of future educationalists includes a course on "methodology". This is at any rate the custom in our country but, judging from the literature in general, this brain-washing policy is an international feature. Please imagine a student of mathematics, of physics, of - let me be cautious, as I am not sure how far this list extends - impregnated, in any other way than implicitly, with the methodology of the science that he sets out to study; in any other way than by having him act out the methodology that he has to learn! In no way do I object to a methodology as such - I have even stimulated the cultivation of it, but it should be the result of *a posteriori* reflecting on one's methods, rather than an *a priori* doctrine that has been imposed on the learner.
>
> I readily admit that the principle of "learn first, apply later" works in educational methodology no better than it ever did in mathematics; that is, where it works it does so to the benefit to a small minority of learners only - the future specialists of methodology. Yet, fortunately, the intimidated majority can count on the precious assistance of this authoritative guild, the pure methodologists, whose strength consists in knowing all about research and nothing about education. (Freudenthal, 1991, pp. 150-51)

Whilst Freudenthal is harsh on 'pure methodologists', his suggestion that the researcher should experience an intuitive sense of 'rightness' in her/his methodology, as an outcome of implicit "impregnation", was a guiding principle at various times when I deliberated how best to proceed with both collection and interpretation of data. The interplay between action and reflection has a place in the progress of knowledge and understanding of all kinds. Justification of methodology, in particular, may be the outcome of reflection on research activity.

INTERPRETATION AND PARTICIPANT OBSERVATION

In particular, an interpretivist perspective on the data caused me to revise my initial view of my influence on and contribution to the data I had collected. In a summary of some characteristics of qualitative research, Burgess (1985, p. 8) suggests that "studies may be designed and redesigned":

> All the methods associated with qualitative research are characterised by their flexibility. As a consequence researchers can turn this to their advantage, as a rigid framework in which to operate is not required. Researchers can, therefore, formulate and reformulate their work, may be less committed to perspectives which have been misconceptualised at the beginning of the project and may modify concepts as the collection and analysis of data proceeds. (*ibid.*)

Perhaps the shift of 'perspective' which most complicated my interpretation of the rich transcript data that I was obtaining was my eventual acceptance that my contribution to the clinical interviews was itself an object of interest. Hence the kernel of my attention, which initially was on the nature of children's mathematical constructs, shifted towards interaction, language and affect. This was, at first, a shift which I made with some reluctance. It initially came about because, when I made public presentations of my work, my talk would be about the *children*, the subjects of my interviews, with reference to fragments of transcripts which I had distributed. Invariably, before long, someone would turn the discussion to some aspect of my contribution as interviewer or 'teacher'. [Note 0.9] It seemed pointless to deny that my part in the conversation had some influence on the child's, yet I had begun determined to be neutral in these clinical interviews, a mere channel (as it were) for the outpouring of the child's mathematical thinking. If that had been my expectation, it was not confirmed by the peer group feedback I received from these seminars.

My recognition of my influence as teacher-substitute, simultaneous with my role as clinical interviewer/researcher, was double-charged. On the negative side, I could no longer claim to be eliciting some kind of 'pure' cognitive data from these children. Yet Piaget had never suggested that one could, and acknowledged the influence of the interviewer on the subject, at the very least as a person to be 'satisfied' with answers - Piaget (1929, p. 16) speaks of children "romancing", sometimes inventing plausible, supposedly-introspective accounts in order to produce an answer to the interviewer's question. On the positive side, I no longer had to defend my clinical interview technique as a flawless, quasi-psychiatric performance, but was at liberty to consider it critically as a quasi-pedagogic transaction insofar as it succeeded in managing the children's interaction with me and (in the study reported in Chapter 5) with each other, enabling children to predict, generalise and explain *and* to articulate their beliefs. Furthermore, if I analysed some transcripts as if they were teacher-pupil interactions (and not just investigator-subject interviews), then there would be greater prospect of drawing conclusions that could be relevant to other teachers with regard to their pedagogic interactions with pupils.

I do not, however, believe that this shift of perspective is in conflict with the standard guidance to clinical interviewers to resist the urge to teach. The shift of perspective came in the analysis of the transcripts, not in the method which guided my conduct of the interviews. *The point is not that I viewed myself as teacher in these interviews, but that the children may have.*

The first phase of this study, the weakly-framed interviews with Susie principally, and also with Simon, correspond to my period of strongest 'denial' of my influence in the interaction, and here (Chapter 4 and Appendix 2) the cognitive dimension in my analysis and interpretation of the children's language is strongest.

In the next phase Chapter 5) I begin to consider the subtle ways in which children convey uncertainty, distancing themselves from their assertions in which they lack confidence. This is a feature of the interaction between the child and the interviewer perceived as teacher-substitute. And the 'teacher' himself (me) is also seen to use hedges and other indirect language, as a mark of respect for the children's 'face', and to sustain their active participation. The effort to interpret my own linguistic behaviour (in response, initially, to the urging of others) is a dangerously schizophrenic enterprise, and has to be recognised as just that. The major justification is that I have some access to the interviewer's intentions in any utterance that he makes. That, of course, depends on memory and integrity - adults are capable of 'romancing' too. At least my interactions with these children were recorded and transcribed to become, as I remarked in Chapter 0, a permanent object in the world, detached from the event. This transformation is helpful, but the major safeguard for me was (again) peer evaluation: between 1991 and 1995 I regularly shared my progress with research students and supervisors at research days at the Centre for Mathematics Education at the Open University, gave nine accounts of aspects of my work at conferences and research seminars, and enjoyed the critical attention an 'Informal Research Group' (this is reported in Chapters 7 and 8).

Next, I designed a developmental study (Chapter 6) and interpreted data collected from it. This study also involves interaction between children and an interviewer, although I was not involved in the interaction myself beyond the pilot stage. Finally, all but one of the transcripts which I analyse in Chapter 7 do not involve me, being records of other teachers teaching. Here I am not, in any sense, a participant observer; I view these cases as events from the outside, events which confirm the widespread occurrence of vague features in mathematical teacher-pupil interaction, and give me confidence to interpret related aspects of these interactions. The exception, in that Chapter, is my mathematical conversation with a student. Here at last I can present myself, no longer as split-personality teacher/researcher, but simply as a teacher enjoying mathematics talk with a student.

In effect, then, I begin as 'pure' contingent-questioner and end as 'pure' teacher. The fixed point, for the purpose of analysis and interpretation of texts, is that I must be researcher throughout. What I cannot claim as fixed is my perspective on my role - the study had to be "designed and re-designed" (Burgess, *op. cit.*, p.8). But a changed perspective is a fresh insight; in that sense at least I am encouraged to cling to Freudenthal's precarious but liberating proposal (1991, p. 150) that methodology might be created in and identified from action; that is that "methodology [...] should be the result of *a posteriori* reflecting on one's methods".

SUMMARY

This thesis will encompass a number of themes and related aims, with the superordinate aim of revealing and analysing some of the ways that participants in mathematics talk use vague language to achieve their communicative and affective purposes. Much of the transcript data that I analyse is obtained from contingent, clinical interviews with children, which may be described as mathematical conversations. I have reported a shift of perspective in my analysis of the transcripts, (though not in the method which guided my conduct of the interviews) as a consequence of acknowledging that my contribution to the conversations must be one factor to be taken account of in the analysis. Whereas I did not view myself as teacher in these interviews, it is possible that the children may have.

One tenet which I held constant before, during, and after the study for this thesis, is a belief in the central place and function of generalisation in mathematics. In consequence, this process is central to the design and analysis of much of the thesis. I therefore devote the next chapter to an examination of the ways that generalisation gives rise to the greatest delight and satisfaction in mathematical activity, and also to the greatest uncertainty.

CHAPTER 1: GENERALISATION

> Analysis and natural philosophy owe their most important discoveries to this fruitful means, which is called *induction*. Newton was indebted to it for his theorems of the binomial and the principle of universal gravity. (Laplace, 1902, p. 176)

> I have had my results for a long time, but I do not yet know how I am to arrive at them. (Gauss, quoted by Lakatos, 1976, p. 9)

Much of the empirical work to be reported in this thesis is set in contexts where students (of various ages) are carrying out and talking about mathematical tasks. The precise mathematical content of the tasks is of less importance, for my study, than the mathematical processes which the students are engaged in. In order to understand what students say in such circumstances, and why they say things the way they do, it is important to understand the nature of the processes themselves.

Yet, it can be dangerous and unhelpful to make sharp distinction between process and content in mathematics. Arguably it is the content - numbers, shapes, groups, topological spaces, and so on - that most clearly distinguishes mathematics as mathematics, that marks it out from other domains of knowledge, for example science or history - whilst in both of these cases there are content overlaps with mathematics. Time, for example, is a concern for all three. On the other hand, without the processes there would be no mathematics, or at least mathematics would have no products, no propositional content, no truths (theorems) about the objects of study - numbers, groups, and so on. Bell et al. (1983, p. 206) describe the process dimension of mathematics in terms of:

> the style and atmosphere of the activity in the mathematics classroom [...] whether [pupils] see mathematics as a field of enquiry, or a deductive system, or a set of methods to be learnt from the teacher.

It is the *activity*, as opposed to the product of the activity, that picks out process aspects of mathematics. Shuard (1986, p. 104) lists a number of aspects of mathematics which were considered to be processes by a group of primary school teachers. It is a long list, and begins (gratifyingly, from the point of view of subsequent themes in this thesis) with: classifying, generalising, predicting and estimating. The list concludes with some processes related to personal qualities: cooperating, working independently, persevering. Shuard comments:

> It is interesting to note that these processes are all couched in terms of 'doing'; we *do* a process [...] processes are actions, or verbs. Some of these verbs, however, have related nouns which represent areas of mathematical content that express the same ideas. (p. 105)

The glib answer to the question "What is mathematics?" is "Mathematics is what mathematicians do". Leaving aside the inherent circularity of this statement, it is a definition that stresses the fact that mathematics is brought into being by human activity.

Processes which feature significantly in this thesis, especially in Chapters 4 to 7, are generalisation, prediction, explanation and estimation. The extended discussion of generalisation which follows will comprehend significant (for this study) aspects of prediction and explanation, and have unexpected links with estimation.

GENERALISATION AND INDUCTIVE REASONING

To begin, here are two tasks. [Note 1.1] Each proposes some activity - things to do - and poses a question or questions - things to think about in consequence of the activity.

Task 1. Partitions.

> The integer 3 can be 'partitioned' into an ordered sum of (one or more) positive integers in the following four ways: 3, 2+1, 1+2, 1+1+1. Find all such ordered partitions of 4. In how many ways can other positive integers be partitioned?

Task 2. Reflections.

> Draw two intersecting lines l, m in the plane. Choose a motif M (such as a capital F) and position it in the plane. Locate in turn the image M' of M under reflection in l, and the image M" of M' under reflection in m. How is M" related to M? Name a (composite) plane transformation which maps M to M". What happens if you choose other pairs of lines, other motifs and initial positions?

Suppose, then, that I (a student) carry out these tasks: I produce the activity, and consider the questions. Having done the tasks, I might ask myself the questions spontaneously, even had they not been explicitly stated - an act of curiosity as a consequence of the activity. The activities themselves yield 'data', secure but isolated 'knowledge', information-in-hand. The questions (spontaneous or explicit) cause me to look beyond the isolated items of information, to view that information as known samples from a (substantially) unknown class of phenomena, as specific instances of items in that class. The questions associated with the tasks suggest limitations to the class, although I may choose to work with restrictions or extensions of such limits.

In the case of the given Task 1, my information-in-hand will soon include the 4 given partitions of 3, the 8 possible partitions of 4, and very possibly the 2 partitions of 2. The question "In how many ways can other positive integers be partitioned?" prompts the thought that my data belong to a class $P=\{(n,r(n)):n\varepsilon N\}$ where $r(n)$ is the number of partitions of the natural number n. My data consist only of the subset corresponding to $n=2,3$ and 4. What might the (infinite) remainder of the class be like? The information-in-hand is limited, and in any case is bound to be insufficient in itself for me to *know* the values of $r(n)$ for values of n beyond those for which I have data. It may, however, be sufficient to cause me to form some beliefs about those unknown $r(n)$ values, and I may be prepared to articulate such beliefs in terms of predictions or conjectures.

For example, I may observe that $r(n+1)=2r(n)$ for n=2,3 and hence predict that $r(5)=2r(4)$ i.e. $r(5)=8$ on grounds of cognitive systematization (Rescher, 1979). Such a prediction is amenable to *confirmation.* In this case, to achieve confirmation I would need to undertake identification and listing of the partitions of 5. Once confirmed, I might then go on to predict values of r(6) and r(7). I may anticipate the corresponding acts of confirmation with eagerness, or with distaste as they become progressively (in this case, exponentially) more tedious. I may use a computer to automate the listing of partitions. All this will add to my sense of regularity in the system, but will always leave a countable infinity of unknown values of $r(n)$.

I may, indeed I am likely to, go further than making a finite (in principle, confirmable), set of predictions, and make the following conjecture: that for *all* natural numbers n, $r(n+1)=2r(n)$. Since I know that $r(2)=2$, I may formulate the conjecture as: for all natural numbers n, $r(n)=2^{n-1}$. These conjectures have the quality of generalisations; they are statements (of beliefs) about properties of an entire class, statements made despite the fact that the whole class has not been directly inspected and tested - indeed, *could* not be - for the property or properties in question.

Prediction can be viewed as a specialized form of generalisation. Each feeds on the other, each is both parent and child of the other, although predictions are (generally) more straightforward to articulate since they entail fewer quantifiers. As to the nature of mathematical generalisation in terms of cognitive activity, an individual observes events, instances of some kind in a mathematical domain, of numbers or shapes perhaps. Some compelling desire to "make sense" in a holistic way, of this set of information inputs - to impose regularity, to gain predictive power - seems to drive an involuntary unifying tendency, a generalising force of the intelligence. One of the rewards of maturity can be delight in the awareness of insight. It is an intense *physical* sense of well-being, of things holding together, that I have committed to public writing only twice (Rowland, 1974; 1993). In a recent e-mail conversation, Anne Watson (1995) attempts here to say what generalising *feels* like:

> I only know what it feels like to *me* when I generalise, I cannot say what it feels like to someone else, but I assume the mental action of generalising feels like something to other people. I found it very difficult to explain "how to generalise" in words or actions when I was a classroom teacher, but quite effective to catch a moment when generalising seemed to be going on and suggest that children tried to hang on to the feeling of that moment, whatever it was, so that they might recognise it again in future. For me, I feel a different level of power, an approaching completeness, a different positioning of self when I generalise but I bet this won't be a universally useful description.

The kind of generalisation I have described in the context of Task 1 is a classical 'pattern spotting' activity (Hewitt, 1992) - duly, meaninglessly and sometimes joylessly performed by the nation's adolescents since GCSE coursework institutionalised mathematical 'investigations' (Love, 1988, p. 250). The object of this kind of pattern spotting is to identify some mapping whose domain is the natural numbers.

Task 2: Reflections, is not of the same kind, for in this case a generalisation associates a plane transformation with each pair of lines in the plane. The initial position of the motif affects its images, but the composite transformation, the 'product' of the two reflections, is independent of that position. That observation is itself a generalisation; another generalisation following from it might be a claim that (for intersecting lines) the composite of two reflections is always a rotation. At another level, the angle of the rotation can be related to the angle between the lines l and m. The associated class from which information-in-hand is available then consists of triples $(l,m,t(l,m))$ where $t(l,m)$ is the composite of reflections in l and m. The imaginative problem poser is aware of how the class can be extended; the judicious one knows about closing down parameters in order to highlight regularities.

Generalisation is a particular form of the epistemic phenomenon of *induction* [Note 1.2], and is properly and usefully considered from that broader perspective. The term 'induction' is derived from the Latin rendering (using *ducere*, to lead) of Aristotle's *epagoge* (*epi-agoge*, leading outside). The change of prefix, from out(side) to in(side) is interesting; inductive reasoning takes the thinker outside the evidence, by somehow discovering (by generalisation) some additional knowledge inside themselves. The mechanism which enables an individual to arrive at plausible, if uncertain, belief about a whole population, an infinite set, from actual knowledge of a few items from the set, is mysterious. The nineteenth-century scientist William Whewell captures the wonder of it all:

> Induction moves upward, and deduction downwards, on the same stair [...] Deduction descends steadily and methodically, step by step: Induction mounts by a leap which is out of the reach of method. She bounds to the top of the stairs at once [...] (1858, p. 114)

Here deduction is portrayed in terms of descent, just as the argument or syllogism is presented on the written page - methodical, steady, safe, sterile, descending. By contrast, induction is framed as daring, creative, ascending. Whewell discusses the complementary characters of induction and deduction (or 'demonstration'), and the symbiotic relationship between them. They must be "processes of the same mind". Without induction there is nothing to justify by demonstration; but it is the business of deduction to "establish the solidity of her companion's footing". [Note 1.3] We may describe the process of inductive reasoning and theorise about it, but it remains a mystery that an individual may confidently claim an infinite kind of knowledge from a finite kind of information base. The notion of 'knowledge of variability' (Holland *et al.*, 1986), to be considered later, offers some insight into the conditions which seem to trigger and assist such inductive leaps into that which is unknown, yet which may be grasped with sufficient confidence to allow articulation.

WHAT IS INDUCTIVE REASONING?

The philosopher Nicholas Rescher (1980) offers some ways of looking at inductive reasoning that have relevance for subsequent chapters, especially Chapters 5 and 7. He begins by emphasising that the crucial thing about induction is its movement *beyond the evidence* in hand. It is a tool for use by finite intelligences, a solution to the problem of *providing answers to questions* on the basis of limited evidence.

I suggest that questions requiring inductive solutions, such as "What is the composite of any two given reflections?" *must* in principle be infinite in character. That is to say, the set of instances of the phenomenon which are the subject of the question must in principle be an infinite set to require an inductive solution. For if the set were finite, it would (in principle) be possible to compute every instance (member of the set) individually, thereby acquiring certain knowledge as to whether in every case such instances conformed to some supposed rule or regularity.

Rescher (1980, pp. 8-9) goes on to analyse possible responses to the question:

[Q] "Is it the case that every F is also a G?".

Mathematical examples include: Are all prime numbers odd? Is n^2+n+41 prime for all $n \varepsilon N$? Is the angle subtended at the circumference of a circle by a diameter always a right angle? Are all cyclic groups abelian? [Note 1.4] Rescher leaves aside the motivation for asking such questions, i.e. the process by which they are generated as matters worthy of attention. The argument which follows is based on Rescher's epistemological analysis.

Suppose we observe a finite set of Fs. Suppose further that each observed F is indeed a G. On this evidence, we are bound then to agree that at least *some* Fs are Gs. Moreover, we have no evidence as yet that there exists any F which is not a G. We could just say "I don't know" in response to Q. Whilst this is truthful, it is also evasive, adding nothing to either knowledge or belief. In the circumstances the response "Yes, all of them are" intuitively presents itself as the *best available*, albeit provisional response; the optimal solution from the point of view of plausibility. I use 'plausible' here in the way that Polya uses the word - not in the sense of being specious, but of being pleasing, satisfying. (Latin *plaudere*, to applaud, clap hands).

Thus, induction can be seen to represent a responsible form of cognitive 'gap-filling', which supposes that our consistent sample of Fs faithfully represents the whole. This is not to claim that the solution "Yes, all of them are" securely represents the truth, but that it qualifies as the *best estimate to the truth* which we are able to make, on the basis of the evidence available.

An inductive inference can be viewed as an aspiring but failed deductive inference, in the following sense. Suppose I set out to examine the claim (C, say) that no integer strictly between 1329 and 1360 is prime (Watson, 1994). This is equivalent to the conjunction of thirty individual propositions $\{P_i\}$ where P_i asserts that i is composite, and i takes every integer value between 1330 and 1359. From thirty premises P_{1330}, P_{1331}, P_{1332}, ..., P_{1359}, I am entitled to infer C, because it is a syntactic conclusion of these premises. Suppose I then proceed to test the truth of each member of the set $\{P_i\}$, for example by noting the divisor 2 for even values of i and then running divisibility checks for each of the ten possible odd prime divisors between 3 and 31 for each remaining odd value of i. I confirm that P_{1330} is true, P_{1331} is true, P_{1332} is true, ...,

P_{1359} is true. Since classical semantics is faithful to syntax, I am now assured of the *truth* of C. I have given a secure, deductive demonstration of it, in the form:

P_{1330}
P_{1331}
P_{1332}
.
.
.
P_{1359} (premises above the line)

———

C (conclusion below the line)

In contrast, consider now the situation with inductive inference, typified by (Task 1):

The number of partitions of 2 is a power of 2

The number of partitions of 3 is a power of 2

The number of partitions of 4 is a power of 2

The number of partitions of 5 is a power of 2

The number of partitions of 6 is a power of 2

<The number of partitions of any integer greater than 6 is a power of 2>

———

The number of partitions of every positive integer is a power of 2

The first five premises can be directly confirmed as true (by listing and counting), but do not by themselves justify the conclusion below the line. The plausible inference of the conclusion is *enthymematic* (information extending); that is to say, the missing but necessary premise (shown inside the brackets < >) is tacitly supplied, thus presenting an inductive argument as if it were a deductive one. Since enthymematic premises are normally suppressed, there being no firm evidence on which to claim their truth, the induction has the appearance of a failed (incomplete) deductive inference. The missing premises are tacitly supplied in order to enable us to cross the 'epistemic gap' which separates the data from the 'answer' (to essentially infinitary questions). The epistemic gap is the 'residual distance' to be accomplished, requiring nothing less than an 'inductive leap'. As I have already argued, the inductive conclusion represents the most satisfying solution, the most plausible truth-estimate available, and so provides a *post facto* justification of the tacit addition of enthymematic premises.

This is not to say that the inductive conclusion has the status of certain knowledge. Nor is it simply an uninformed guess. It is a *conjecture*. As Polya says, discussing

Goldbach's (unproved) conjecture that every even non-prime is a sum of two primes:

> We arrived so [from a finite data set] at a clearly formulated general statement, which, however, is merely a conjecture, merely *tentative.* That is, the statement is by no means proved, it cannot have any pretension to be true, it is merely an attempt to get at the truth. The conjecture has, however, some suggestive points of contact with experience, with "the facts", with "reality". It is true for the particular even numbers 10, 20, 30, also for 6, 8, 12, 14, 16. (Polya, 1954, p. 5)

Polya's reference to an "attempt to get at the truth" shares the same epistemic quality as my earlier reference to induction as truth-estimation in the erotetic enterprise.

Observe that inductive inference is to be distinguished from stochastic (statistical or probabilistic) inference. Given that every F we have examined is also a G, I do not conclude (inductively) that most Fs are Gs. The inductive inference that *every* F (including the infinite set of F's which we have not inspected) is a G, is an uncompromised, if provisional, commitment to a regularity in 'nature'. The degree of commitment of an individual to the inductive conclusion, the extent to which they *believe* it to be true, may and does vary considerably.

At the heart of this thesis is the study of how individuals are able to convey, by spoken language, the strength or fragility of inductive truth-estimates. A link can, and will, be made with conventional estimation of quantities. Indeed, Chapter 6 focuses on vagueness in the context of children giving a plausible estimate of the size (cardinality) of a discrete set of objects. The notion of induction (of which generalisation is one form) as truth-estimation forges the following pleasing association between generalisation and estimation. Given a discrete set, there exists a precise integer n_0 such that the statement "there are n objects in the set" is a true statement when $n=n_0$; for all other values of n the statement is false. An estimate of n_0 is therefore, in common with an inductive inference, a pragmatic, optimal solution to an erotetic dilemma. In the case of estimation, an accepted, associated language of approximation exists, including words like 'around' and 'about'. This language is cultivated to express the awareness of the speaker that, whilst the claim that s/he makes is an *estimate* of the truth, in the classical bivalent sense (if $n \neq n_0$) it may actually be false. In the case of generalisation, I will show that speakers find their own pragmatic, linguistic means of conveying such awareneses.

INDUCTION AND THE MIND

A cognitive account of inductive reasoning, and generalisation in particular, should include both a description of mental re-structuring to include the acquisition of new knowledge (or a new way of looking at old knowledge) and an attempt to explain how the individual accomplishes the 'inductive leap' as a synthetic act. The literature seems to be strongest in the former aspect, and much of it relates in one way or another to the formation of concepts.

Abstraction

Skemp (1979) gives an account framed in broadly Piagetian terms. We possess cognitive schemas (networks of concepts) of various kinds which structure our perception of reality; in fact, they sensitise us to reality, but in a selective way. For example, I have a schema which includes the concept 'polygon' and links it to a number of spatial, numerical and aesthetic concepts. This schema has a selective influence on the way that I perceive spatial inputs, in that I am able to process them comfortably within the schema that includes 'polygon', and I will do so if at all possible. This schematic shaping of reality (or whatever we choose to call the incoming data) is 'assimilation'. But the concept 'polygon' came about (for me) by a process of 'abstraction' (*ibid.*, p. 24) so that I might include objects like pentagons and hexagons (less familiar) in a conceptual class along with triangles and quadrilaterals (more familiar). Note that inclusion of certain objects entails exclusion of others; concept formation requires that certain qualities be stressed whilst others are ignored.

One consequence of this generalisation (to the concept 'polygon' from a number of instances) is to bring within my reflective horizon an infinite class of concepts (including 22-gons and 469-gons) examples of which I have never seen, nor am I ever likely to. Nevertheless I am able to state and prove theorems about such objects. Skemp calls such generalisation (concept expansion) 'reflective extrapolation'. It has the quality of Piaget's notion of accommodation, which Skemp prefers to call 'expansion'. Thus, Skemp speaks of assimilation *by* a concept, and expansion *of* a concept.

It is certainly a feature of inductive reasoning that the truth of an infinite (enthymematic) set of untested propositions is claimed, in order to expand and bind together a finite (usually small) set of items of data. The essential finiteness of the data-base of information-in-hand may be obscured by the manner in which it is obtained and presented. I am thinking here of the new generation of dynamic geometry software, typified by *Cabri-Géomètre*. Suppose, for example, that I create a triangle and construct its three medians. I observe that the medians are concurrent. I vary the triangle by dragging one of its vertices on the screen. The concurrence of the medians is an invariant of every frame in the cinematographic presentation. The inductive generalisation is readily made, and with conviction (Schumann and Green, 1994, pp. 85-6); perhaps because the software has enabled a vast set of confirming instances to be realised. Indeed, given the apparent continuity of the dragging process, the data set appears to be continuous, uncountable. This is an illusion, since the hardware design - pixels and the like - only permits a finite, though vast, set of configurations to be calculated and displayed.

Construction and Expansion

Harel and Tall (1991) observe that 'generalisation' may refer both to a process (inductive thinking) and to the product (an inductive inference) of that process. They distinguish three different kinds of generalisation:

- expansive generalisation, the expansion of a schema without need for its reconstruction;
- reconstructive generalisation, which occurs when a subject reconstructs an existing schema in order to widen its range of applicability;
- disjunctive generalisation, the construction and addition of a new, disjoint schema to an existing one, to deal with a new context.

Rather than give Harel and Tall's example (of three students solving linear equations), I prefer to cite the growth of my own understanding of two ideas from group theory. Let H be a subset of some group G, and C1 the idea that I can calculate the cosets of H in G. Let C2 be the idea that it is possible to calculate the conjugacy classes of the group G. Having first encountered C1 (for the purpose of proving Lagrange's theorem on subgroups), I subsequently met C2 (perhaps in order to enumerate normal subgroups, I don't really remember). Initially I regarded C2 as a concept disjoint from C1. Both C1 and C2 were set in a schema of related examples and theorems - a case of disjunctive generalisation . Somewhat later I took an interest in equivalence relations, was struck by the beauty of the Fundamental Theorem (that the equivalence classes induced by an equivalence relation form a partition), and became aware of the unifying significance of the set partition theorems associated with both C1 and C2. I could, moreover, write down equivalence relations R1 and R2 on the elements of G which induced the respective C1- and C2-partitions of G. Insofar as I now perceived C1 and C2 as being examples of the same thing, this was expansive generalisation. Many years later, in the study of transformations of vector spaces $V_n(F)$, I learned about group actions (in order to prove the orbit-stabiliser theorem). The partitions associated with C1 and C2 were both, I deduced, the sets of orbits of elements of G under suitably-defined actions of G on itself. This reconstructive generalisation, I felt, not merely included C1 and C2, but made them somehow inevitable and special (in both senses of the word) cases. A group action will partition *any* set that it acts upon - including the group itself.

Disjunctive "generalisation" barely merits the name at all, since it misses the opportunity for economy of intellectual effort and places a heavy load on memory. As Polya puts it:

> there are two kinds of generalisations, one is cheap and the other is valuable. It is easy to generalise by diluting; it is important to generalise by condensing. (Polya, 1992, p. 11)

Harel and Tall argue that expansive and reconstructive generalisations are "more appropriate for cognitive development", and that expansive generalisation is the more straightforward of the two. I would comment that, in the context of inductive activity and learning, disjunctive generalisation could relate to the accumulation of individual but isolated instances of a phenomenon; expansive generalisation has some quality of conservative extrapolation, rather like a prediction of the next case on the basis of the previous cases; reconstructive generalisation seems like inductive inference, so that each observed instance is viewed as a special case of a phenomenon with wide applicability. Harel and Tall conclude:

> In principle we believe that the most desirable approach to generalization is to provide experiences which lead to a meaningful understanding of the current situation, to allow the move to the more general case to occur by expansive generalization, but there are times when the situation demands a re-construction and, in such cases, it is necessary to provide the learner with the conditions in which this reconstruction is more likely to take place. (p. 39)

It may be fair to observe that mathematics education lacks a unified *theory* of generalisation, although most mathematics educators will say they know it when they see it.

Intuition

How is it that humans (and other animals) manage to organise experience in such a way as to provide a basis for judgement about situations outside experience? The mysterious nature of this capability is nicely captured in the word 'intuition' - that which (at first, or never) we cannot rationalise, we label intuition. The Latin root of the word means 'to look inside', suggesting in-tuition, or teaching of/by the inner self. Fischbein (1987) uses 'intuition' to denote a "type of cognition" by which we recognise some 'facts' about the world. Fischbein emphasises a distinction between intuition and perception, the later being awareness of objects and facts as a result of sensory inputs. Knowledge (more correctly, beliefs) gained as a result both of perceptions and intuitions may be false. For example, visual perceptions permit optical illusions; naive intuitions are the basis of laws of 'intuitive physics' such as "velocity is proportional to applied force". (Orton, 1985; Champagne *et al.*, 1980, p. 1077)

It is worth adding that intuition can be an obstacle to the acceptance of truth; for example (Fischbein's), it is virtually impossible to accept intuitively that a set may be equivalent to one of its subsets (Dedekind's characterisation of an infinite set). We exploit this - at least, I do - to play linguistic tricks on our students when we demonstrate that $n \rightarrow 2n$ defines an injection of N, and then ask them to agree that there are "as many" even natural numbers as there are natural numbers. Tirosh (1991, p. 203) tested the plausibility of this particular infinite-cardinal 'paradox' and several others with a sample of 1381 students aged 11-17, finding 'incorrect' intuitions commonplace in the comparison of infinite sets. What is particularly interesting is her finding that student misconceptions were relatively stable across this age range. One of the great didactic challenges (not just with regard to intuitions about infinity) for mathematics education is the identification and confrontation of students' 'epistemological obstacles' (Bachelard, 1938; Cornu, 1991) - beliefs which become embedded in a knowledge-schema because they function well in one domain of activity, but which malfunction and lead to contradictions in another.

Intuition can be viewed as a form of induction, in the sense that:

> intuition [...] always exceeds the given facts. An intuition is a theory, it implies an extrapolation beyond the directly accessible information [...] *One may affirm then that intuitions refer to self-evident statements which exceed the observable facts.* (Fischbein, 1987, pp. 13-14)

It appears that intuition can be said to occur when an individual reaches a conclusion on the basis of less explicit information than is ordinarily required to reach that conclusion. (Westcott, 1968, p. 97)

In everyday use we perhaps stretch this to a point beyond induction, so that intuition, rather like premonition, advises us of facts and events in the absence not just of adequate evidence, but of any evidence whatsoever. Such intuitions may be ascribed to 'extra-sensory' forces and powers - in effect, extending the scope of the notion of sensory inputs, so that intuition becomes 'merely' a different kind of perception. By a criterion of inner confidence in the intuitive disclosure, however, this kind of cognitive experience seems not to be excluded by Fischbein:

> An intuition always exceeds the data on hand. However, being an extrapolative guess is not sufficient to define an intuition. A feeling of certainty is also a necessary characteristic of an intuition. Otherwise *it is a mere guess [...] The extrapolativity aspect is not always evident, because the apparent obviousness of intuitions hides the incompleteness of the information on which they are based.* (p. 51)

One could read the italicised (by Fischbein) comment in two ways. The 'extrapolativity' may not be evident in the (intended) sense that the individual may be unaware that the basis of evidence for their new 'knowledge' is incomplete, that it is indeed a generalisation. Alternatively, in the case of 'mere' everyday intuition, and possibly some cases of mathematical intuition, the extrapolativity may not be evident in the sense that the individual may be unaware that the new knowledge builds in some way on old experience; s/he may not consciously be aware of some deeply embedded kinds of evidence that s/he does in fact possess.

Constraints and Default Hierarchies

Peirce (1932) noted the capability of humans to exercise appropriate, pragmatic constraints in the kinds of questions that they ask, so as to gain information which is not just new, but (in some sense) worth knowing. He poses the question as to how individuals manage to ask the 'right' questions about the data (the fruits of experiences of every kind) available to them.

> Nature is a far vaster and less clearly arranged repertoire of facts than a census report; and if men had not come to it with special aptitudes for guessing right, it may well be doubted whether [...] their greatest mind would have attained the amount of knowledge which is actually possessed by the lowest idiot. (paras 2:752-3, pp. 474-476).

The matter of constraints is also a factor in the calculation of how much data we need before a generalisation can appropriately be made. Suppose (Holland *et al.*, 1986) I visit a remote Pacific island. I see a bird which my informant calls a "shreeble", and it is blue. I am likely to suppose (a conjecture, provisional generalisation) that (all) shreebles are blue. Suppose, however, that I see an inhabitant of the island whom my informant calls a "Barrato", and he is obese. I am not likely to suppose that all Barratos are obese. Why the difference? As John Stuart Mill put it,

> Why is a single instance, in some cases, sufficient for a complete induction, while in others myriads of concurring instances, without a single exception known or presumed, go such a very little way towards establishing a universal proposition? (1873, p. 314).

The answer proposed by Holland *et al.* lies in the notions of (a) default hierarchies of concepts, and (b) variability of objects with respect to others in the default hierarchy. In the example above, my default hierarchy recognises the set of shreebles as a subcategory of the class of birds, and BLUE as a member of the category of colours. The observation of one blue shreeble activates the possibility that all shreebles are blue. I then calculate (i.e. make a judgement about) the extent of *variability* of the superordinate category BIRD with respect to the superordinate property COLOUR, by reference to my knowledge of appropriate subordinates - robins, ravens, seagulls, parrots, and so on. In this case, my calculation suggests only modest variability in the BIRD-COLOUR relation, and so I attach some confidence to the conjecture (the possibility activated in my mind) that all shreebles are blue. A more sophisticated judgement might result from a more refined default hierarchy, e.g. choosing TROPICAL BIRD as the immediate superordinate category to SHREEBLE. The difficulty with the obese Barrato should now be evident: body-shape (from skinny to obese) varies considerably within peoples of any given nationality - at least, there is sufficient variability for me to be unwilling to make any conjecture about Barratos from such a modest base of evidence. Indeed, it is unlikely that the observation of the single example (the obese Barrato) would even trigger the suspicion, the generalised conjecture.

Stamp (undated) recalls teaching a lesson on right-angled triangles. In the first two examples considered - (6, 8, 10) and (5, 12, 13) - it was observed that the area and perimeter had the same numerical value. This led to the conjecture that "this happens every time". Stamp reports that he "denied" that this can be so, and in fact proceeds in the note to deductive demonstration that, with the exception of the given examples, the proposition is universally false! Why, given two confirming instances, was Stamp disinclined to formulate the conjecture? What prompted him to spontaneously denial that it could be true? The default hierarchy/variability analysis is certainly plausible here; mathematics teachers are very conscious of the confusion between perimeter and area, and very aware themselves that there is considerable variability between the two in fact. Had the class tried just one more right-angled triangle they would have been obliged to modify their hypothesis. One can speculate that anxiety that a false relationship might be inferred for *all* triangles or polygons explains Stamp's 'authoritarian' intervention.

What we believe or judge to be significant as a basis for generalisation, in given circumstances, enables a distinction to be made between the logic and intuition of inductive confirmation. Consider the hypothesis: "All ravens are black". The observation of a black raven is clearly a confirming instance, strengthening conviction that the proposition "All ravens are black" is true. But the hypothesis is *logically* equivalent to its contrapositive, which says that all non-black things are not ravens. Thus, on this logical account, it follows that a white shoe (for example) is a confirming instance, confirming and adding to belief that all ravens are black. Yet, intuitively, it does no such thing. This dilemma is called "Hempel's paradox" (Hempel, 1965). The approach to generalisation through default hierarchies and variability accounts for, and justifies, the intuitive, sceptical reaction. For whereas (as with shreebles) we are in a position to judge the variability of the superordinates BIRD and COLOUR, the concepts NON-BLACK and NON-RAVEN have no meaningful superordinates, and we cannot even begin to make variability judgments.

MATHEMATICAL HEURISTIC

George Polya must take the credit for a revival of interest in mathematical heuristic in the two decades following the second world war. Heuristic, by Polya's definition (1945, p. 102) is the study of methods and rules of discovery and invention. Heuristic reasoning, he affirms, is in the service of discovery, and so is to be regarded as provisional and plausible; it is often based on induction, or on analogy. Polya codified heuristic methods and strategies; for example - if you cannot solve the proposed problem, can you imagine a more accessible related problem? (*ibid.*, p. 103). Polya's *How to Solve it*, published in his 58th year, has all the freshness of a rediscovered art. This is still possibly the best-known of his heuristic quintet (Polya 1945; 1954a and 1954b; 1962 and 1965). Unfortunately, it has been widely (mis)represented, in summary, as a four-stage *recipe* for problem solving:

- understanding the problem
- devising a plan
- carrying out the plan
- looking back

as if adherence to some heuristic algorithm were a sure path to a solution. The four-point list is is too general to be helpful in the solution of actual mathematical problems; it is, I suggest, more realistically viewed as a generic analysis of problem solving rather than a prescriptive guide to action. I concur with Burton (1984, p. 10) that problem solving cannot be taught, although one can over time acquire a few techniques (such as tabulating data) and a great many, less tangible heuristic instincts (such as calling to mind familiar, analogous situations; see below).

Perhaps Polya wants to correct the false impression that the processes of mathematical problem solving can be "learned" when later, an old man, he writes:

> Solving problems is a practical art, like swimming, or skiing, or playing the piano: you can learn it only by imitation and practice. [...] if you wish to learn swimming you have to go into the water, and if you wish to become a problem solver you have to solve problems. (Polya, 1962, p. v)
>
> If he [the problem solver] possessed a perfect method, an infallible strategy of problem solving, he could determine the next step from the data of the incoming situation by clear reasoning, on the basis of precise rules. Unfortunately there is no universally perfect method of problem solving, there are no precise rules applicable to all situations, and in all probability there will never be such rules. (*ibid.*, p. 89)

It is clearer in *Induction and Analogy* that what Polya is doing is encouraging the problem solver to recognise and reflect on their own heuristic, as well as that of some distinguished mathematicians of the past, such as Pappus, Descartes, Leibnitz, Euler, Laplace, Bolzano.

> I tried to illustrate each important point [...] in several cases I was obliged to take a not too elementary example to support the point impressively enough. In fact, I felt that I should present also examples of historic interest, examples of real mathematical beauty ...
>
> I should add that for many of the stories told the final form resulted from a sort of informal psychological experiment. I discussed the subject with several classes, interrupting my exposition frequently with such questions as: "Well,

> what would you do in such a situation?" ...
>
> In short, I tried [...] to give an appropriate opportunity to the reader for intelligent imitation and for doing things by himself. (Polya, 1954, p. vii)

Polya goes on to stress the central place of induction as a paradigm for plausible reasoning, and continues:

> Observe also (what modern writers almost forgot, but some older writers, such as Euler and Laplace, clearly perceived) that the role of inductive inference in mathematical investigation is similar to its role in physical research. Then you may notice the possibility of obtaining some information about inductive reasoning by observing and comparing examples of plausible reasoning in mathematical matters. And so the door opens to *investigating induction inductively.* (p. viii)

Thus, *Induction and Analogy in Mathematics* is a primer for the reflective, mathematical fieldwork that the reader is to undertake; it supplies "the data for the inductive investigation of induction" which is to come in *Patterns of Plausible Inference.* In the same way, *Mathematical Discovery, Volume I* (1962) is the reader's mathematical preparation for the cognitive exploration in *Volume II* (1965). This is the way of the inductive investigation of induction, and it is the method for the reflective investigation of mathematical problem solving. As William Whewell put it:

> For an Art of Discovery is not possible. At each step of the investigation are needed Invention, Sagacity, Genius - elements which no art can give. We may hope in vain, as [Francis] Bacon hoped, for an Organ which shall enable all men to construct Scientific Truths [...] this cannot be. The practical results of the Philosophy of Science must be rather *classification and analysis of what has been done,* than precept and method for future doing. (Whewell, 1858, p. v)

TRUTH AND CONVICTION: THE THEORY OF NUMBERS

The Theory of Numbers is notorious as fertile ground from which to generate inductive inferences; those (like Goldbach's conjecture) which deny as yet any counter-example, yet defy deductive proof, have a way of acquiring celebrity status. Perhaps Fermat's Last Theorem is currently the best-known, given the interest in Andrew Wiles' 1993 endeavours (Granville and Katz, 1993).

Polya (1954, pp. 91-98) gives, *in extensio,* a translation of a memoir of Leonard Euler.

It is an account by Euler of his discovery of a 'formula' - a recursive scheme, in fact - to determine the sum $\sigma(n)$ of the divisors of (in principle) any positive integer n. Interest in $\sigma(n)$ derives, in part, from Greek fascination with 'perfect' numbers, such as 28 and 496, for which $\sigma(n)=2n$. My purpose in re-examining the story is to gain insight into the nature of belief and conviction, from the perspective of a genius, to use an inadequate cliché, among mathematicians.

Euler begins by remarking on the lack of orderliness in the sequence of prime numbers, and proceeds to an exposition of the well-known method of evaluating $\sigma(n)$ from the prime factorisation of n, using the simplicity of $\sigma(p^k)$ for primes p and the multiplicative property of the function σ. He then lists the calculated values of $\sigma(n)$ for n=1 to 99:

1 3 4 7 6 12 8 15 13 ... 120 252 98 171 156

Euler remarks that the list, like the sequence of primes, is a disorderly one (arguably more so, since it isn't even monotonic). Nonetheless he continues, "I just happened to discover an extremely strange law" to bring order to the apparent chaos. The 'law' is what we would call a recursive one, defining $\sigma(n)$ in terms of the set of preceding values $\sigma(m)$, for $m<n$. At first sight the scheme is as chaotic as the sequence it purports to explain:

$$\sigma(n) = \sigma(n-1) + \sigma(n-2) - \sigma(n-5) - \sigma(n-7) + \sigma(n-12) + \sigma(n-15) - \sigma(n-22) - \sigma(n-26) + \sigma(n-35) + \sigma(n-40) - \ldots \text{ taking, if necessary, } \sigma(0) \text{ to be } n.$$

On inspection, the aspect of the scheme which is most obscure is the sequence 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, ... Euler explains that it will become clear by listing first differences: 1, **3**, 2, **5**, 3, **7**, 4, **9**, 5, **11**, 6, **13**, 7 ... The patterns in the even and odd (placed) terms of this sequence is now evident, so that the list is indefinitely extendable.

There is no sign as to how Euler arrived at his recursive scheme by inductive consideration of the data. Indeed, it is not clear from the memoir that he did; it appears that he derived it from yet another of his number-theoretic conjectures, in the theory of partitions. In any case, he had no deductive proof that the law is universally true (i.e. for all $n \in \mathbb{N}$). Euler freely admits this to be the case; the quotations which follow are from Polya (1954, pp. 93-95):

> I must admit that I am not in a position to give it a rigorous demonstration ...

It is now fascinating to examine what Euler considers sufficient grounds for belief in the generalisation - for the reader and, presumably, for himself, since he has sufficient confidence to publish the result. He proceeds:

> it is not difficult to apply the formula to any given particular case, and so anybody can satisfy himself of its truth by as many examples as he may wish to develop.

That is to say, the finite set of confirming instances can be as large as the reader chooses, and anybody can satisfy "himself" of their truth. He cannot, of course, establish (in a demonstrative sense) the truth of the formula (the generalisation) by pointing to any number of examples, except by the addition of enthymematic premises. Euler, by reference to truth *by*, rather than *of*, examples, is claiming that (despite his admission that he cannot 'demonstrate' the truth of the formula) it will be possible - in this case at least - to achieve *conviction* of its truth. Not content to leave it to the reader to generate some data, he says:

> I will justify it by a sufficiently large number of examples.

Clearly "justify" must be about plausibility rather than proof, if Euler is not to be accused of 'naive empiricism' (Balacheff, 1988, p. 218). What is "sufficiently large" is a tricky question, but Euler implicitly suggests that 20 belongs (in this context) in the vague category of sufficiently large numbers, since he proceeds to recursive calculation, with his formula, of the values of $\sigma(n)$ for $n=1$ to 20. In every case the value coincides with that given by direct calculation by means of prime decomposition. Was he just lucky? Certainly not:

> I think these examples are sufficient to discourage anyone from imagining that it is by mere chance that my rule is in agreement with the truth.

"In agreement" - the "rule" and the "truth" coincide for these 20 values of $\sigma(n)$. Yet awareness that, for example, n^2+n+41 is prime for $n=1$ to 39, but not prime for all $n \varepsilon \mathrm{N}$, can bring about a sceptical frame of mind, which Euler recognises. In particular, the recursive calculation of $\sigma(20)$ only calls on six previous terms; $\sigma(n-1) + \sigma(n-2) - \sigma(n-5) - \sigma(n-7) + \sigma(n-12) + \sigma(n-15)$. He acknowledges the tentativeness behind his remark "I think these examples are sufficient":

> Yet someone *could still doubt* whether the law of the numbers 1, 2, 5, 7, 12, 15 [...] is precisely that one which I have indicated [...] Thus the law could still appear insufficiently established and, therefore, *I will give some examples with larger numbers* (emphasis added).

He then picks 101 and 301 (the first prime, the second composite) and proceeds to confirm, as it were, agreement between the rule and the truth. Surely, he seems to say, that clinches it:

> The examples I have just developed will undoubtedly dispel any qualms which we might have had about the truth of my formula.

Here Euler nicely exemplifies the role of *prediction* in relation to conjecturing activity. Its effect is to dip into the box of enthymematic premises, pick an item, and to examine *either* (a) whether it is in agreement with, i.e. an *extrapolative extension* of, the finite set of data-in-hand, or (b) whether it is a *confirming instance* of an already-formulated generalisation. The more random the choice, the more powerful is the epistemic effect of the confirming instance. It is like the celebrity drawing the winning raffle ticket, who looks away from the box as she dips her hand into it, or the magician who rolls back his sleeves to show that nothing is concealed from the audience. The "choice" of 101 and 301 is most interesting. With a "live" audience, Euler could invite numbers to be tested; in written exposition he must appear to pick them randomly. In effect, he is saying, "If it works for 301 it must work for anything". The psychological thrust of this was recently illustrated for me by some first year undergraduate mathematics students investigating the continued fractions of $\sqrt{n}$ for $n \varepsilon \mathrm{N}$. Their findings were written up in project reports.

Emma notices that $\sqrt{3} = [1, \overline{1, 2}]$, $\sqrt{6} = [2, \overline{2, 4}]$, $\sqrt{11} = [3, \overline{3, 6}]$, $\sqrt{18} = [4, \overline{4, 8}]$. She makes a conjecture (inductive inference), tentatively expressed, and a conservative extrapolative prediction:

> Whenever n is of the form r^2+2 it seems as if the continued fraction is always of the form $[r, \overline{r, 2r}]$. Therefore I would predict that $\sqrt{27} = [5, \overline{5, 10}]$.

Emma proceeds to confirm her prediction.

Sarah generates data similar to Emma's, and makes the conjecture that $\sqrt{(n^2+1)} = [n, \overline{2n}]$ for all $n \varepsilon \mathrm{N}$. Her prediction then proceeds:

> I then picked a large value of n to check this general solution *to make sure it worked.* (emphasis added)

She predicts that $\sqrt{82} = [9, \overline{18}]$ and proceeds to confirm it. For Sarah, the case $n=9$ serves to confirm, to make sure of, all remaining instances. She suggests, as Euler might have done, that this is an example which will undoubtedly dispel any qualms that we might have had about the truth of her formula. Emma proceeds cautiously, her prediction being for the next value (5) of n outside her data. There is a strong sense here that she needs to reassure herself first, before she considers convincing others. I shall analyse my own experience of the need for such personal reassurance towards the end of this chapter.

Both students conformed to the normative requirements of proof, by algebraic transformation of their generalised surds, using the continued fraction algorithm. But Sarah, in common with Euler, offers us a 'crucial experiment' (Balacheff, 1988, p. 218) in preparation for (Euler - in place of) the deductive argument. The crucial experiment tests the plausibility of a generality by confirmation of an instance which is *chosen* for being *not-special*; a nice paradox.

I shall weave some personal remarks into a summary of the remainder of Euler's memoir. For me the Theory of Numbers has been an inner laboratory, more like a playground, in which I experimented and theorised as an adolescent, read as a sixth former (when I first discovered the existence of books on primes, divisibilty and the like), studied as an undergraduate (personally supervised by Keith Hirst, since there was no taught course), and latterly returned to as a teacher. The course which I give (in its content and the *inductive* approach *to* the content) is inspired by, and based on, Burn (1982). The course includes a topic on partitions (unordered, in distinction to Task 1). The sequence $\{p(n)\}$ of partitions of 1, 2, 3, 4 ... is in fact 1, 2, 3, 5, 7, 11, 15, 22, 30, ... The list has something of the impenetrable randomness of the sequence of primes. The course includes a Lemma:

> *$E(n)$, the number of partitions of n into an even number of distinct parts is equal to $O(n)$ the number of partitions of n into an odd number of distinct parts, unless n is of the form $\frac{1}{2}m(3m\pm1)$; that is to say, unless $n=$ 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, ... ; in which case $E(n)$ and $O(n)$ differ by 1.*

Whilst I had not encountered Euler's formula for $\sigma(n)$ before I read Euler's memoir, I certainly recognised the sequence 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, ..., although I could not imagine such a direct connection between $\sigma(n)$ and $p(n)$. This situation exemplifies the unconscious activation of one element of Polya's heuristic catechism :

> Here is a problem related to yours and solved before. Can you use it? (1945, p. 19)

In my case, the answer was "No". It was by analogy (with the Lemma on partitions) that I recognised the obscure pattern of Euler's sequence, yet I was unable to develop the analogy in order to achieve any explanation of Euler's formula for $\sigma(n)$. As Pimm observes (1981 and 1987, p. 100), analogy assumes a recognition of *similarity* (Greek *analogia* - 'proportion') between two situations, but has a preferred direction of application. My knowledge of p(n) offered potential for the illumination of $\sigma(n)$, even if the potential was, for me, unrealised.8

It turns out, however, that Euler had deployed proto-Polyan heuristic, and arrived at his "extremely strange law" for $\sigma(n)$ precisely because he had become aware of such a similarity. He did, after all, invent the theory of partitions. The memoir concludes with an ingenious but 'formal' proof of the 'law'. (Encumbered by our modern inhibitions about convergence, its validity would trouble us no end.) Assuming [Note 1.5] that the generating function s for E(n)-O(n) is $(1-x)(1-x^2)(1-x^3)(1-x^4)$...., Euler demonstrates that $-(x/s)\, ds/dx$ is the generating function for $\sigma(n)$, and thus forges the link between $\sigma(n)$ and p(n). The remainder of Euler's argument is unimportant for my present purposes, the study of awareness of mathematical process.

EXPLANATION AND PROOF

In mathematics, proof may fulfil, at any time, one or more of a number of purposes. These purposes include not only assurance of truth, but explanation of (accounting for) observed regularities; and clarification of what it is that is being claimed (Hersh, 1993).

For example, consider Task 1: Partitions.

Seeking to account for the observation that the number of (ordered) partitions of each positive integer is twice that of the previous one, I argue as follows. Consider any partition of n. If I increase the *size* of the last part by 1, I have produced a partition of $n+1$. If instead, I adjoin an *additional* part of size 1, I have produced a second partition of $n+1$. So there are at least twice as many partitions of $n+1$ as there are of n. Finally, I have to make some remarks about there being no duplicates and no partitions of $n+1$ unaccounted for. This constructive argument is very satisfying in that it explains why this remarkable and unexpected doubling phenomenon occurs. It is then easily adapted to a proof by Mathematical Induction that, for all $n \varepsilon$N, $r(n) = 2^{n-1}$.

Later, I have an entirely different insight. Imagine n as a sequence of n 1's, separated by n-1 boundaries (one boundary between each consecutive pair of 1's). Each partition of n can be achieved by removing some of the boundaries, and "gluing together" (adding) the 1's which are no longer separated by a boundary. For each of the n-1 boundaries I have two options: remove it or leave it in place. Therefore there are 2^{n-1}

partitions of n.

Polya (1954, p. 114) remarks that this "happens not infrequently", i.e. that a theorem which is proved first by Mathematical Induction is subsequently proved "by some other method". I would comment that my second proof is "neat", economical, yet it leaves me little wiser about *why* I got the doubling pattern. Of course, I realise that 2^n is indeed double 2^{n-1}, but that observation seems to have no contact with the original problem about partitioning integers. This is at the heart of Hewitt's (1992) complaint that "their [children's] attention is with the numbers and is thus taken away from the original situation".

The first argument above (relating partitions of $n+1$ back to those of n) is effectively presented, from the point of view of concreteness and conviction, by assigning a particular value to n, say 4. The exposition then describes how each partition of 4 begets two partitions of 5. Indeed, my experience with students indicates that careful scrutiny and comparison (with $n=3$, say, for manageability) of the 4 partitions of 3 alongside the 8 partitions of 4, can trigger explanatory insight concerning the way each partition of 3 is related to two partitions of 4. Such an argument amounts to proof by 'generic example' (Mason and Pimm, 1984; Balacheff, 1988).

> The generic proof, although given in terms of a particular number, nowhere relies on any specific properties of that number. (Mason and Pimm , 1984, p. 284)

The story (probably apocryphal, but see Polya, 1962, pp. 60-62 for one version) is told about the child C. F. Gauss, who astounded his village schoolmaster by his rapid calculation of the sum of the integers from 1 to 100. Whilst the other pupils performed laborious column addition, Gauss added 1 to 100, 2 to 99, 3 to 98, and so on, and finally computed fifty 101s with ease. The power of the story is that it offers the listener a means to add, say, the integers from 1 to 200. Gauss's method demonstrates, by generic example, that the sum of the first $2k$ positive integers is $k(2k+1)$. Nobody who could follow Gauss' method in the case $k=50$ could possibly doubt the general case. It is important to emphasise that it is not simply the fact that the proposition that the sum $1+2+3+ \ldots + 2k = k(2k+1)$ has been verified as true in the case $k=50$. It is the *manner* in which it is verified, the form of presentation of the confirmation.

By contrast, consider the (false) proposition that n^2+n+41 is prime for all $n \varepsilon \mathrm{N}$. I may confirm the truth of the instance when, for example, $n=30$; by evaluating $30^2+30+41$, which is 971, and checking that no prime from 3 to 31 divides 971. But this gives no insight whatsoever as to why n^2+n+41 is prime for any other value of n. (The ambiguity of 'any' suits me well here; I want it to mean first 'some' (true) and then 'all' (false).) By contrast, I could demonstrate, with a diagram, that $5^2+2x5+1$ is a perfect square, in a manner that could convince that n^2+2n+1 is a perfect square for all positive integers n.

As Balacheff (1988) so clearly and elegantly puts it:

> The generic example involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of the class.
> (p. 219)

Closely related, if not identical, to proof by generic example, is the notion of 'action proof' (Semadeni, 1984; Walther, 1984). The generic example serves not only to present a confirming instance of a proposition - which it certainly is - but to provide insight as to *why* the proposition holds true for that single instance. Walther indicates, for the validity of an action proof, the psychological necessity of the identification of aspects of special examples which are "*invariant* regarding a transfer to other arbitrary examples". (*ibid.*, p. 10). The transparent presentation of the example is such that analogy with other other instances is readily achieved, and their truth is thereby made manifest. Ultimately the audience can conceive of no possible instance in which the analogy could not be achieved.

In effect, the generic example triggers an *inductive inference*; that the *argument* holds in all cases. In saying this, I am suggesting that the generic example (suitably and skilfully presented) has the same role for proof as the confirming instance does for generating a conjecture. This may appear to be a fundamental methodological flaw in the method of proof by generic example. That is to say, it appears that the existence of a general proof is no more than a conjecture, an inductive inference in fact from the generic example. But this is to ignore the different demands of confirming instance and generic example, in the form of their presentation. The confirming instance only has to be demonstrated to be true, by any means whatever; the subtlety or lack of it in the demonstration is of no consequence. But in the generic example, the demonstration must be more than a demonstration of truth; it must in some way explain, account for the property, in one instance, in the process of confirming it, so that that one instance is seen to be more than a case of serendipity.

The ability, indeed the tendency, of young children to explain by generic example is a feature of Chapter 5 of this thesis. I detect in the 'mathematical community' the general view that such proofs are naive and imperfect. That view is represented by Tall:

> Of course, *it is essential* in advanced mathematics to take the step from [generic] explanation to formal proof. (1991, p. 9, emphasis added)

I question that conventional view, and believe that learners of mathematics at all levels, including university students, should be assisted to perceive and value that which is generic in their particular insights, explanations and arguments. The barrier between such a level of knowing and the writing of "proper" proofs is then seen for what it is - a lack of fluency not with ideas, but with notation.

RECOLLECTION

A thesis is, or can be, a personal document presented for public examination; a marker in time of change in the writer over time. I therefore take the liberty, before concluding this chapter, of reassessing a personal, but publicly-documented event. I believe that I can use it to illustrate and pull together some of the foregoing threads.

In Rowland (1974), I described a sequence of intellectual and domestic events leading to a mathematical insight (about regular polygons and generators of dihedral groups) which caused me very great excitement. It centred on a class of polynomials and their roots. At one point I was stuck with the cubic a^3-5a^2+6a-1. A solution and indeed, a generalisation, derived from the crucial insight that a solution of the previous polynomial, a^2+3a+1, could be written as ψ^2, where ψ is the golden ratio, realised geometrically by the ratio [side of pentagram : side of pentagon].

I offered (p. 46) an 'elegant' geometrical confirmation of the truth of this insight, before announcing the generalisation that it suggested to me. I had no idea as to why the generalisation should be true, I simply dared to hope that it *could* be - not (just) because I wanted to solve the problem that I had set myself, but because *this* solution had such beauty. I specialised the generalisation I had made to three simpler cases, involving a regular hexagon, a square and an equilateral triangle, and was able to confirm them easily. If my generalisation held, then the square of the ratio [side of heptagram : side of heptagon] would satisfy the stubborn cubic.

What I needed, at that moment, was merely to confirm that it did; for, if not, I had a counter-example to my generalisation, and the prospect of disappointment. The momentary prediction required confirmation without delay. In contrast to pentagons and hexagons, I knew nothing of the pure geometry of the heptagon; as I put it:

> Too excited to be sophisticated, I resort to trigonometry. [A diagram of a regular heptagon then indicates the angle θ between a side AB and a short diagonal AC.] $\theta = \pi/7$ so $(AC/BC)^2 = (2\cos \pi/7)^2 = 3.24$ approx., which [...] *does* seem to satisfy my cubic.

Aware of the blunt tools I had used for confirmation, I then asked:

> Can anyone demonstrate this more elegantly?

I was, perhaps, seeking a demonstration that had the quality of a generic example. The rest of the article is concerned with the plausibility of the generalisation, but (at that time) I had no proof. I confessed:

> It has to be admitted that my "conclusion" is rather a sweeping one, being pure conjecture based on six instances of the result. So far I have no proof [...] I need hardly affirm my belief in the conclusion, but as a piece of mathematics the work is incomplete ...

The strength of my "belief", in the total absence of proof, surprises me now. So strong it was, that I was prepared to go into print, to put it on record. The editor, David Fielker, did not hinder me. When a postgraduate student, Alan Barnes, presented me with a proof just as MT69 went to press, I was pleased but not surprised that my belief was vindicated. In the end it was the beauty of the conjecture that assured me - nothing could, at the same time, have such beauty, yet be false.

> Beauty is truth, truth beauty - that is all
> Ye know on earth, and all ye need to know.
> [Keats, 1820, *Ode to a Grecian Urn*]

SUMMARY

For all learners of mathematics there is the possibility of acquiring new knowledge by reflection on appropriate and relevant experience (and arguably there is no other way). Generalisation - unifying and information-extending insight - is central to such a means of coming-to-know, and may be viewed as a form of inductive reasoning. In the introduction to his inductive *Pathway into Number Theory*, Burn (1982) reminds us of an adage of Jacques Hadamard, that the purpose of rigour is to legitimate the conquests of the intuition. For the great mathematicians, as well as for novices, mathematics characteristically comes into being by inductive intuition, not by deduction. The products of induction are plausible truth-estimates. Therefore tentative belief, as opposed to certain knowledge, is an essential component of mathematical thought. In the next chapter I shall link this observation to constructivist and quasi-empirical philosophies of mathematics learning, and begin to explore how vague language can be used to advantage in talk about beliefs and provisional knowledge about mathematics.

CHAPTER 2: PERSPECTIVES ON VAGUENESS

> It is a mark of the educated man and a proof of his culture that in every subject he looks for only so much precision as its nature permits. (Aristotle, *Nicomachean Ethics*)

> Two plus two equals five - for sufficiently large values of two. (*per* Eric Love, source unknown)

The empirical database of this thesis, which will be considered in Chapters 4 to 7, consists almost entirely of transcripts of mathematical discourse. Many of the features of this discourse which are singled out for special attention are aspects of vagueness, as it is manifest in the spoken language. I shall claim that both students and teachers employ and exploit vagueness in mathematical discourse. I mean, by that, that the emergence of vagueness as a surface feature of mathematics talk is not evidence of any linguistic carelessness or deficiency, but that vagueness of various kinds - such as indeterminacy, ambiguity, approximation - is deliberately and explicitly utilised by speakers to achieve particular ends in the context of mathematical discussion, exposition, questioning, and so on. This view is at odds, superficially at least, with a conventional view of mathematics as the language *par excellence* of precision. The purpose of this chapter is to present a background for the subsequent analysis of vagueness in mathematics talk.

PRECISION AND TOLERANCE

The first page of the 1982 Report of the Committee of Inquiry into the Teaching of Mathematics in Schools (the Cockcroft Report) included, in bold type, an assertion that:

> mathematics provides a means of communication which is powerful, concise and unambiguous. (HMSO, 1982, p. 1)

and proposed the communicative power of mathematics as a "principal reason" for teaching it. There was a refreshing novelty in such a claim, which seemed to be justifying mathematics teaching in much the same way that one might justify the learning of a foreign language, and it did much to promote and sustain popular interest in the place of language in the teaching and learning of mathematics. Such a view of mathematics is in contrast, however, with that expressed in a more-or-less contemporary pamphlet issued by the Association of Teachers of Mathematics, whose authors argued that:

> Everyday speech is a highly tolerant medium. This tolerance is necessary because conversation is a form of action in the world; [...] Because it is a tolerant medium, everyday language is necessarily ambiguous.
>
> [...] Now, mathematising is also a form of action in the world. And its expressions, however carefully defined, have to retain a fundamental tolerance [...] Because it is a tolerant medium, mathematics is also necessarily an ambiguous one. (ATM, 1980, pp. 17-18)

This description of mathematics and conversation as forms of action emphasises mathematics as human activity and suggests discourse as a means of communication, for mutual understanding and agreement. Furthermore, it offers the radical proposal that ambiguity is a beneficial ingredient in the formulation, the "expression" of mathematics. As a *product* (polished, final), mathematics may be presented, particularly in writing but also in speech, as though it lacked ambiguity, representing truths about the world - or at the very least, about itself - in a sure, exact and unequivocal kind of way. This is tidy, but it is a deception of sorts. As Goguen puts it:

> Exact concepts are the sort envisaged in pure mathematics, whilst inexact concepts are rampant in everyday life. This distinction is complicated by the fact that whenever a human being interacts with mathematics, it becomes part of his ordinary experience, and is therefore subject to inexactness. (1969, p. 325)

The issue of vagueness has received relatively little attention in the literature of mathematics education, with the exception of the dimension of lexical ambiguity. The language with which mathematics is communicated and shared is a subtle blend of words and syntax, which is rooted in natural language or 'Ordinary English', but which also includes elements of technical language - 'Mathematical English' (Kane *et al.*, 1974) - much of which has Greek or Latin etymology and logical force (e.g. 'or' is interpreted inclusively). In Rowland (1995a) I have analysed this blend, and some of the difficulties experienced by learners, in terms of mathematics talk taking place at two discourse levels - essentially object level and meta-level. The object level language bears the mathematical and logical substance which is capable of being coded in a formal first-order theory of mathematical logic. The meta-level language is not mathematics *per se*, but a means of social interaction between people *doing* mathematics. It is what remains when the object level text is stripped away from the text of discourse.

Here, for example, two eleven-year-olds, Kerry and Runa are finding pairs of integers whose sum is fourteen. The elements of mathematical object langage are shown as [*abc*], and were in fact recorded on paper in symbols by the girls as they talked.

T8:73	Runa	Let's put [fourteen], and then ..
74	Kerry	Right. [Ten add four] Underneath.
75	Runa	[Ten]. OK.
76	Kerry	[Ten add four] um [twelve add two] Um [thirteen add one] ...
77	Runa	Wait a minute.
78	Kerry	[Five. Nine add five]
79	Runa	Yeah, I was just thinking that.

The distinction between the two language levels, object level and meta- level, is often blurred because the mathematical object language incorporates and re-defines many words from natural language. Ambiguities are inherent in words such as 'difference', 'similar' and 'or', whose mathematical meanings are related to their ordinary English meanings, but which cannot be inferred from them with adequate precision; they have to be learned within a process of mathematical enculturation. The clashes and ambiguities which result, especially for learners of mathematics, from this seemingly haphazard appropriation of ordinary English for mathematics is widely appreciated (Ullmann, 1962, Chapter 7; Shuard and Rothery, 1984, Chapter 3; Pimm, 1987, Chapter 4; Barham, 1988; Durkin and Shire, 1991). One strand of the literature (Otterburn and Nicholson, 1976; Hardcastle and Orton, 1993) has highlighted the lexical deficiency of novices as users and and interpreters of mathematics at the object level, whereas I shall explore the interactive fluency of pupils in the use of mathematical meta-language.

Hans Freudenthal (1978, pp. 259-60), whilst not directly acknowledging the significance of vagueness as a conceptual or linguistic phenomenon, is nevertheless critical of careless or spurious precision, and cites a Dutch encyclopedia giving the length and weight of a lion as 2.40-3.30m and 180-225kg - data which betray their non-metric origins. [Note 2.1] In another example, he points out that whereas, with regard to size, there is no significant difference between 10^{10} and 10^{10} -1, there is a world of arithmetical difference, in that one is divisible by 9 and the other is not. The difference between the two situations is *pragmatic,* i.e. related to purpose - the appropriate degree of precision depends on what you want to *do* with the number. Freudenthal points out the need for pupils to acquire judgement to distinguish between two worlds; "the world where precision is a virtue, and the other where it is a vice, and [...] to be at home in both of them".

These essentially pedagogic perspectives on mathematical and linguistic aspects of vagueness can be viewed as contributions to a debate concerning the place of precision, prescription and completion as factors in the acquisition of knowledge in general and mathematics in particular. In the remainder of this chapter, I review some ways in which this debate is rooted in philosophies of mathematics and of language.

VIEWPOINT: MATHEMATICS AND MATHEMATICS EDUCATION

The Cockcroft view of mathematics as precise and unambiguous reflects a popular view of mathematics, based on a tacit 'absolutist' philosophy. That is to say, that the truths (theorems) of mathematics are sharp and certain, and in some way represent objective knowledge. Indeed, in this view, mathematics stands above and apart from empirical science in its purity and freedom from experimental error. Science can only offer 'theories', whereas the objects of mathematical thought and the assertions which are the products of mathematics are certain. Such a view is represented with passion and eloquence by G. H. Hardy in *A Mathematician's Apology*:

> mathematical objects are so much more than they seem. A chair or a star is not in the least like what it seems to be; the more we think of it, the fuzzier its outlines become in the haze of sensation which surrounds it; but '2' or '317' has nothing to do with sensation [...] 317 is a prime, not because we think so, or because our minds are shaped in one way rather than another, *because it is so*, because mathematical reality is built that way. (1940, p. 130)

There is a comfortable sense of certainty in Hardy's words, but it reflect the state of mind of one who already knows. In the next section I present a 'fallibilistic' view of knowledge and truth which offers a truer reflection of the experience of one who is coming to know.

FALLIBILISM

The austere perspective of absolutism, characterised by Hardy, contrasts with and is challenged by a fallibilist philosophy of mathematical knowledge. Fallibilism makes explicit my pedagogical *credo* in this thesis.

First, it is reassuring to note that the rejection of absolutism is neither new nor irresponsible. Absolutism is linked, though not indissolubly, with the platonist's [Note

2.2] belief that mathematical truth is 'out there', pre-existing and independent of human knowledge or lack of it. Each mathematical truth is, as it were, waiting to be discovered by the intelligence who, by genius or diligence, uncovers it. Over the last century, absolutism has been worked out in two major forms, logicism and formalism. The logicism of Russell and Frege attempted to reduce all mathematics to pure logic. Hilbert took the formalist view that mathematics is more than pure logic, but is capable of being axiomatised.

The arguments against absolutism (and, to some extent, against platonism) from within mathematical logic are essentially twofold. First, the deductive arguments which terminate in mathematical theorems must begin from a baseline of axioms, which are plausible products of observation or intuition. Any claim to absolute truth must then be suspect, since the very foundation is beyond the reach of demonstration. Secondly, truth begets truth according to an agreed (or tacit) set of logical axioms and rules of inference. Yet these rules are not beyond question or reproach, and alternatives to the classical scheme (first order predicate calculus) include modal logic (Ackermann,1956) and intuitionist/constructivist logic (Heyting, 1964).

The fallibilist critique of absolutism has been put forward in the writing of Imre Lakatos (1922-1973), notably in his posthumously-published book *Proofs and Refutations* (1976). The book is explicitly set against the background of Polya's mathematical heuristic and Popper's critical philosophy of science (*ibid.*, p. xii). Central to Lakatos' critique is the failure of formalism to account for the growth of mathematical thought, either in peoples (phylogenesis) or in individuals (ontogenesis). Lakatos offers an alternative view of mathematics as the product of human mathematical activity and inter-personal dialogue.

> [...] informal, quasi-empirical mathematics does not grow through a monotonous increase in the number of indubitably established theorems, but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations. (p. 5)

The term 'quasi-empirical' mathematics refers to the observation that conjectures are the inductive outcome of consideration of 'data' collected in mathematical activity. An asymptotic refinement of definitions, theorems and proofs, argues Lakatos, is the outcome of human dialectic, acted out in the histories of cultures, and again (though not necessarily in the same way) in the classroom. In this 'fallibilist' view, mathematics is a relative and subjective form of knowledge, perpetually open to revision.

The fallibilist position has been recognised and adopted subsequently by a number of writers, whose stance is typified by these words of Reuben Hersh:

> It is reasonable to propose a new task for mathematics philosophy: not to seek indubitable truth but to give an account of mathematical knowledge as it really is - fallible, corrigible, tentative and evolving, as is every other kind of human knowledge. (1979, p. 43)

Whilst Hersh's comment is essentially epistemological, Sandy Dawson has explored the profound implications of Lakatos' quasi-empiricist philosophy for the teaching of mathematics. Writing about a "fallibilistic way of teaching", Dawson has recently summarised his insight as follows:

> It was from ideas contained in Lakatos' articles and book that an alternative way of working in mathematics classrooms developed. [...] Lakatos claimed that the creation of mathematics comes about as the result of a process [...] in which a conjecture is created, tested and proved, or refuted and modified, or rejected outright. A classroom designed for pupils to operate in a fallibilistic fashion would provide pupils with a problem about which they could make conjectures as to its solution. [...] Opportunities to test and examine critically each conjecture must also be provided.
>
> A teacher who is functioning fallibilistically [...] establishes a classroom climate in which an atmosphere of guessing and testing prevails, where the guesses are subjected to severe testing *on a cognitive rather than an affective level* [...] where knowledge is treated as being provisional. Because of the provisional nature of knowledge, pupils are encouraged to confront the mathematics, their peer group and, where appropriate mathematically, even their teacher. (1991, p. 197, emphasis added)

Over the last decade, John Mason has, in effect, been a consistent and effective champion of Dawson's pedagogic interpretation of Lakatos' fallibilist philosophy. Writing about the place of conjecturing in mathematical activity, Mason describes the qualities of what he calls a 'conjecturing atmosphere' in which

> every utterance is treated as a *modifiable conjecture*! (Mason, 1988, p. 9; emphasis in original).

Clearly, then, a fallibilist view *of mathematics* has implications for classroom conduct. This is certainly also true of a constructivist view *of learning*, which recognises that knowledge is shaped by individual schemas and social frameworks of thought.

CONSTRUCTIVISM

The precision and tidiness that characterise the public face of mathematics education, as it appears in school mathematics textbooks for example, reflects the sense that the author has made of it, or a sense negotiated and agreed by a group of people. This is not to say that meaning is arbitrary, nor that any meaning is acceptable, but that each individual must construct meaning for themselves. A non-absolutist view of mathematical knowledge is implicit in recent formulations of radical (individual) and social constructivist epistemologies. Both assert the inevitability of the active sense-making role of the individual learner in assigning meaning to mathematical experiences (including the experience of being 'taught' or 'told') which, if not rejected altogether, are assimilated into his or her pre-existing schema, or disturb and cause it to be revised. In this sense, no two individuals can 'know' one thing in quite the same way, although they might both assent to one linguistic or symbolic expression of it. The Piagetian roots of the radical epistemological theory are evident.

> Constructivism is a theory of knowledge with roots in philosophy, psychology and cybernetics. It asserts two main principles [...] (a) knowledge is not passively received but actively built up by the cognizing subject; (b) the function of cognition is adaptive and serves the organisation of the experiential world, not the discovery of ontological reality. (von Glasersfeld, 1989, p. 162)

Radical constructivism shares with Intuitionistic constructivist philosophy a view of the primacy of the individual in intellectual action and construction, and in its apparent neglect (or lack of emphasis) of a cultural dimension to knowledge (Wilder, 1965, pp. 247-8). For a back-to-back discussion of the two 'constructivisms', see Lerman (1989).

The social constructivist account sets individual construction of knowledge in a context of enculturation. Culture (both macro and micro) is the broad *milieu* in which learning takes place and contributes to the framework into which knowledge is integrated. It also emphasises the role of language in the mediation and production of thought and in the development of meaning (Bishop, 1985). In this social sense, it follows that 'negotiated' understandings also possess a certain cultural relativity. There is an imperative in the social account in the particular cultural context of institutional teaching and learning - schools and universities - with affective as well as cognitive implications, such as issues of status and 'face' (Johnson, 1970), dimensions which recur in subsequent chapters of this thesis.

INTERFACE: MATHEMATICS AND LANGUAGE

A central concern for both mathematics and linguistics is the relationship between form and meaning. It ought to follow that insights of each may potentially contribute to the understanding of the other. As an example of the interaction between mathematics and linguistics, one with a formative and recurring influence in this thesis, I now consider a linguistic application, to the vagueness-related problem of 'hedges', of the recent mathematical theory of 'fuzzy sets'.

FUZZY SET THEORY

Classical logic admits only two possible truth values for a statement i.e. 'true' or 'false'. A number of attempts have been made to extend the notion of truth in order to accommodate vague, in-between states and concepts, such as drizzle and adolescence. Peirce had evidently entertained the idea of a three-valued logic (Fisch and Turquette, 1966) and Lukasiewicz (1920) independently developed a triadic logic which attracted some attention.

A more radical solution to the problem of strict semantic interpretation of vague propositions involves a real-valued notion of truth, in which the value of a statement is a real number in the closed interval [0,1]. The truth value of statement A is a measure of the *extent* to which A is true, with 1 and 0 corresponding to perfect truth and falsity.

An early attempt at such a solution was due to the philosopher Max Black (1937). Aspects of Black's approach have been developed more recently by Lotfi Zadeh, in the invention of 'fuzzy set theory' (Zadeh, 1965). An electrical engineer, Zadeh was interested in the design of systems (such as pattern-recognition or air conditioning systems) that 'worked' without requiring unrealistic or prohibitively-expensive amounts of computing power. In other words, he was interested in ways of achieving solutions that are *good enough* as opposed to *exact*. He achieved this through fuzzy set theory, in which elements are deemed to belong to particular sets to a given degree.

Zadeh expounds his theory by reference to the fuzzy set (TALL) of tall men. Suppose we agree that men less than 4ft tall are members of the set TALL with degree 0, and that those over 7ft are tall with degree 1. Zadeh then proposes a smooth curve (more an ogive than a straight line) joining (4,0) to (7,1). The degree of tallness of any person (well, any man ...) can now be read from the graph. Zadeh then extends this to set algebra; if T' is the complement of T and m(T, x) is the extent of membership of x of set

T, then m(T', x) is defined to be 1-m(T, x). Likewise m(T∪S, x) is defined to be max{m(T, x), m(S, x)}. The valuations of all other Boolean functions of T, S, follow from these definitions; an example is given below.

Zadeh's original paper was on fuzzy sets. Four years later, Joseph Goguen had worked out a corresponding fuzzy logic (1969), in which a statement A may be true, false or partially true, insofar as it is assigned a value ('degree') [A] in some partially-ordered set (L, <), the simplest example being the interval [0, 1].

For a thorough (and thoroughly pretentious) survey of the theory and practice of fuzziness, see Kosko (1994). Seven years after Zadeh's initial 1965 paper, Lakoff published a linguistic application of the theory to hedges.

HEDGES

Some recent approaches to the problem of vagueness within the field of linguistics originate in consideration of the meaning and function of a class of words and phrases called 'hedges', which turn out to be central to my interpretation of vague aspects of mathematics talk in this thesis. Hedges include words such as 'sort of', 'about', 'approximately' - words which have the effect of blurring category boundaries or otherwise-precise measures - as well as words and phrases such as 'I think', 'maybe', 'perhaps', which hedge the commitment of the speaker to that which s/he asserts.

The work of Zadeh (1965) and Goguen (1969) laid the foundation for fuzzy interpretation of vague language, and some details of the edifice were worked out soon after in an important paper 'Hedges: a study in meaning criteria' by George Lakoff (1972, 1973). The paper is an ambitious fusion of mathematical logic and linguistics. The result of the 1972 version, presented to the Chicago Linguistics Society, was to import vagueness from the domain of logic and philosophy of language to that of professional linguists.

Lakoff's paper is chiefly concerned with vagueness as it applies to category membership, and addresses the concern that "natural language concepts" such as 'tall' lack sharply-defined boundaries. He makes this point by reference to work by the psychologist Eleanor Rosch, who asked subjects to rank a number of creatures as to the degree to which they matched a prototypical ideal of 'bird'. A well-defined hierarchy emerged, in which robins were seen as typical birds, eagles less so, chickens somewhat less, followed by penguins, with bats hardly at all, and cows not at all. Lakoff gives a resumé of Zadeh/Goguen fuzzy theory, and concludes that:

> one need not throw up one's hands in despair when faced by the problems of vagueness and fuzziness. Fuzziness can be studied seriously within formal semantics [...] For me some of the most interesting questions are raised by the study of words whose meaning implicitly involves fuzziness - *words whose job is to make things fuzzier or less fuzzy.* I will refer to such words as 'hedges'. (1973, p. 471, my emphasis). [Notes 2.3, 2.4]

It is clear from some of Lakoff's examples - e.g. 'sort of', 'in a manner of speaking' - that "words" in the definition is intended to include phrases. Note also that words that "make things [...] less fuzzy" are included by Lakoff in the category 'hedge'; examples include 'typical', 'definitely'. [Note 2.5] Brown and Levinson (1987, p. 145) observe that this sense is an extension of the colloquial sense of 'hedge'; in fact, this sense will feature very little in my pragmatic analysis of mathematics talk.

Lakoff's paper belongs to the linguistic tradition of 'truth-conditional semantics', the purpose of which is to determine the conditions under which a sentence is 'true' (as opposed, conventionally, to 'false'). However, the "formal semantics" which Lakoff has in mind entails specifying conditions under which vague propositions could be said to be true *to some extent.* That extent is measured on a continuum from 0 (perfectly false) to 1 (perfectly true). Lakoff describes a precise valuation of vague predicates and develops a corresponding truth-valuation of propositions through an exact mathematical calculus of truth-degrees - a task that had been set in train by Zadeh.

Truth-degrees need first to be assigned to a set of atomic statements such as 'Jack is tall', 'a penguin is a bird' and 'a rhombus is a sort of rectangle'. Once fixed, the valuation of any composite statement is determined in a precise and non-negotiable way. For example, if Jack is rich to degree 0.7 and handsome to degree 0.4, then 'Jack is rich and not handsome' is true to degree 0.6 precisely. [Note 2.6]

Whilst much of Lakoff's paper is taken up with technical details in mathematical logic, he begins from and frequently returns to the issue of the meaning of vague language *in use.* My subsequent linguistic studies in pragmatics - how speakers and writers use language to achieve their practical purposes - grew out of the root source of this paper. In Chapter 5, I shall draw on a study which categorises hedges, and which I review at this point.

A Taxonomy of Hedges

Hedges can be usefully viewed as one of four basic types. This observation was initially made in a study (Prince, Frader and Bosk, 1982) of paediatric clinicians, whose spoken language in case-conferences turned out to be unusually rich in hedging - about one hedge every 15 seconds. The following representative examples of physician-physician talk (*ibid.*, p. 85) have an authentic ring to them:

> Well, I think he's uh - I think he's always se - I still think he's seizing a- a little bit.
>
> There is evidence that's been presented that makes me think that it might be a little risky.

To elucidate the ways (identified by Prince *et al.*) that different hedges work, I shall introduce and illustrate the four types by reference to this corpus of physician-talk, and also to some intuitive language data.

The first major type of hedges - a **SHIELD** - is exemplified above by "Well. I think that ... " and "There is evidence that's been presented ....". These indicate some uncertainty in the mind of the speaker in relation to some proposition. The marker (such as *I think that*) lies outside the proposition itself, which may be unequivocal. For example the sentence

I:2.1 Maybe the pharmacy is still open [Note 3.1]

invests all the vagueness in the speaker's uncertainty, as opposed to any possible degree of openness of the pharmacy. The speaker is asserting a proposition (call it S) :

I:2.2 the pharmacy is still open (S)

S is thus made available to others, who may then (if they so wish) discuss whether or not it is true, and to act on it if , for example, they are in need of aspirin. The effect of the hedged assertion "Maybe S" is to comment on the plausibility of S without qualifying S itself. This is an important distinguishing feature of Shields in relation to mathematical discourse. For example, with reference to a Pythagorean triple (x,y,z) with $x^2 + y^2 = z^2$

I:2.3 I think that *x or y is a multiple of 3*

The italicised part is a mathematical sentence in the object language (here, a subset of mathematics called Number Theory), whereas the hedge phrase "I think that" is in a meta-language (English: the object-meta distinction is considered further towards the end of Chapter 3). This difference of linguistic status can be made more visible in the form

I:2.4 I think that $[3|x \vee 3|y]$

Such a hedge presents a mathematical assertion in the form of a conjecture, and implicitly invites comment on the conjecture.

With a little fine-tuning, Prince *et al.* subdivide Shields into two kinds. The first of these is termed a **Plausibility Shield**, typified by 'I think', 'probably' and 'maybe'. A Plausibility Shield 'implicates' (i.e. infers, by a mechanism to be discussed in the next) a position held, a belief to be considered - as well as indicating some doubt that it will be fulfilled by events, or stand up to evidential scrutiny.

The second kind, an **Attribution Shield**, implicates some degree, or quality, of knowledge to a third party. A favourite Attribution Shield with the clinicians, with evident attendant suspicion, was "According to the mother ... ". An Attribution Shield may even fail (for whatever reason) to divulge the source or informant, as in the following notice (I have emboldened the hedge-preface) published in March 1996 by the Library Syndicate of the University of Cambridge:

> The Library Syndicate is concerned at the increase in the number of cars parked outside the designated spaces at the front and side of the University Library. [...] **There is evidence to suggest that** the shortage of spaces is exacerbated by people using the Library car park when they are not in the Library.

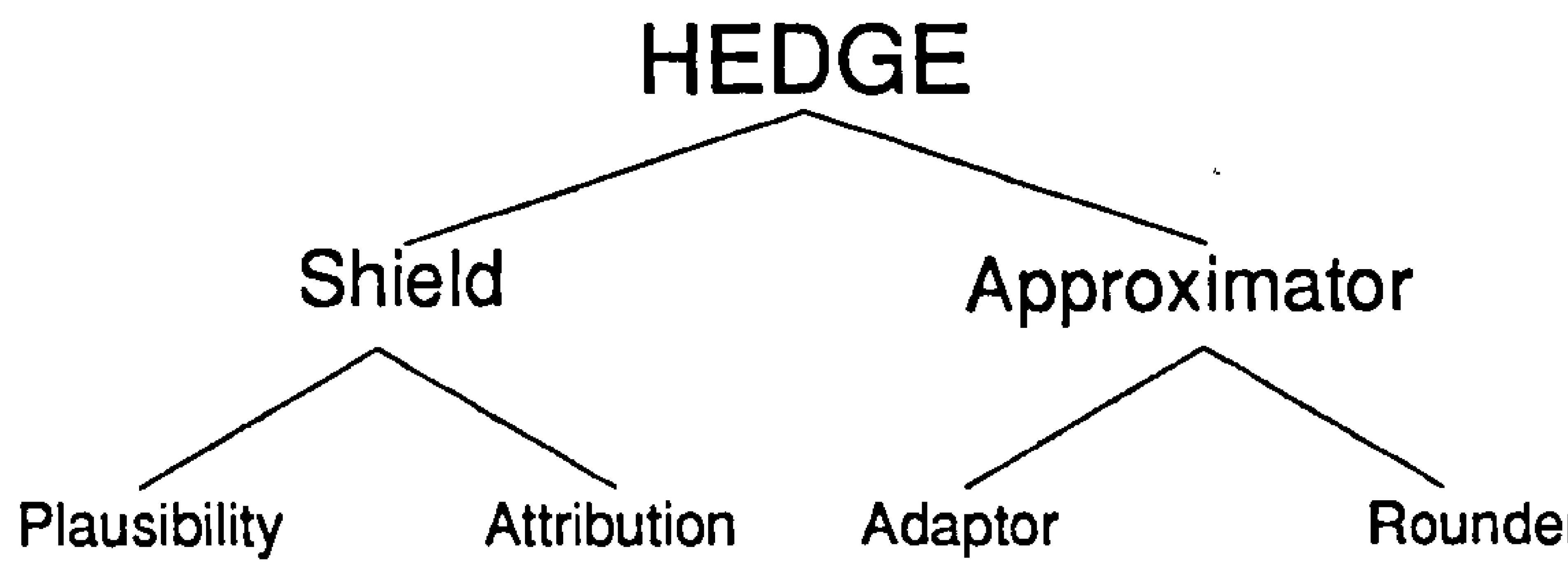
The second major category of hedges (**APPROXIMATORS**) includes 'about' and 'a little bit'. In distinction to Shields, these Approximator-hedges are located inside the proposition itself. The effect is to modify (as opposed to comment on) the proposition, making it more vague. For example, from the Prince *et al.* corpus

> Um, the baby's blood pressure on the ride over here was also about uh something between forty and fifty palpable. (1982 p. 87)

A sub-category of Approximators - called **Rounders** - consists of the standard adverbs of estimation, such as 'about', 'around' and 'approximately', which are commonplace in the domain of measurements, of quantitative data.

The second type of Approximator is called an **Adaptor**. These words or phrases such as 'a little bit', 'somewhat', 'sort of', attach vagueness to nouns, verbs or adjectives associated with class membership. These Adaptors exemplify the hedges which are the subject of Lakoff's semantic work (Lakoff, 1972), and the issue here is class membership.

Prince *et al.* summarise their analysis in the form of the following binary tree:



In Chapter 5, I shall examine the particular purposes achieved by these categories of hedge in mathematics talk. In particular, I shall show that, on occasion, speakers use Approximators for Shield-like purposes.

VIEWPOINT: PHILOSOPHY OF LANGUAGE

Tthe contribution of philosophy to the pragmatic understanding of language cannot be overlooked.

> [when] linguistic pioneers such as Ross and Lakoff staked a claim on pragmatics in the late 1960s, they encountered there an indigenous breed of philosophers of language who had been quietly cultivating the territory for some time. (Leech, 1983, p. 2)

Philosophers had made the problem of vagueness very much part of their territory from about 400 BC, in disputes and attempted resolutions concerning a type of paradox called 'sorites', which means 'the heap'. [Note 2.7]

The first definition of vagueness is, in fact, due to the philosopher-mathematician Peirce, in a dictionary entry:

> A proposition is vague when there are possible states of things concerning which it is intrinsically uncertain whether, had they been contemplated by the speaker, he would have regarded them as excluded or allowed by the proposition. By intrinsically uncertain we mean not uncertain in consequence of any ignorance of the interpreter, but because the speaker's habits of language were indeterminate; so that one day he would regard the proposition as excluding, another as admitting, those states of things. Yet this must be understood to have reference to what might be *deduced* from a perfect knowledge of his state of mind; for it is precisely because those questions never did, or did not frequently, present themselves that his habit remained indeterminate. (1902, p. 748)

Peirce's definition is not easy to penetrate without reference to Peirce's (somewhat impenetrable) semiotic theory, but can be seen to involve (a) modality, uncertainty concerning "possible states of things", (b) the question of borderlines between what is and what is not of a given kind, and (c) inconsistency of speaker habit.

Peirce later theorises about the nature of vagueness in his 1905 paper 'Issues of Pragmaticism' (reprinted in Peirce, 1934), making a distinction between two kinds of indeterminacy; *generality* and *vagueness* (*ibid.*, para. 5.447). The indeterminacy of the former lies in the fact that it refers, not to this or to that, but to anything (in a given class). The indeterminacy of the latter has more to do with class boundaries, so that its field of reference is indeterminate. This distinction will become significant in considering indeterminacy with regard to the referents of pronouns (in Chapter 4).

Peirce acknowledges the endemic presence of vagueness in everyday discourse:

> In another sense, honest people [...] intend to make the meaning of their words determinate [...] they intend to fix what is implied and what is not implied. They believe that they succeed in doing so, and if their chat is about the theory of numbers, perhaps they may. But the further their topics are from such precise, or 'abstract' subjects, the less possibility is there of such precision of speech. In so far as the implication is not determinate, it is usually left vague; (1934, para. 447).

Whereas Peirce is concerned only with vagueness and propositional meaning, my interest extends to propositional attitude and the social functions of vagueness. I will therefore want to go further than Peirce, to claim (e.g. in Chapter 7, Case 8) that vagueness has an essential communicative function in "chat [...] about the theory of numbers".

As I noted in the preface, Peirce is the originator of the philosophical position called 'pragmatism'. His successors include Dewey (1923), Rorty (1980) and Bernstein (1983), who refute objectivism and subscribe to a view that all knowing involves interpretation. The pragmatic tradition locates thought and enquiry in argument and the development of sound judgement (Giarelli, 1988, pp. 23-4). A fallibilist view of *mathematical* knowledge can clearly be seen to flow out of this philosophical stream (Ernest, 1991, p. 201).

Vagueness is also recurrent theme in the philosophical work of another mathematician, Bertrand Russell, from 1913 to 1948. His main ideas are assembled in the1923 paper 'Vagueness', in which he holds that vagueness is not inherent in "things", but is a property of the symbols (including words) that represent them. Things are what they are; both vagueness and precision are features of their representation. He goes on to argue that all language is vague. For example, the word 'red' is vague because

> there are shades of colour concerning which we shall be in doubt whether to call them red or not, not because we are ignorant of the meaning of the word "red", but because it is a word the extent of whose application is essentially doubtful. (1923, p. 85)

Russell makes an elaborate case for the vagueness of all words, including names and even logical connectives, and this lexical vagueness in turn infects all propositions. One vague word is enough to entail the vagueness of a sentence. Russell argues that all knowledge is vague, and our communication of knowledge by language is contaminated by vagueness. In adopting such an extreme position, it is as if Russell is engaging in a philosophical game. However, by exposing us to the thought that everything is vague, he raises our awareness of the possibility of vagueness when we may may least expect it. Moreover, whilst asserting its inevitability, Russell explicitly acknowledges a positive epistemic characteristic of vagueness which will feature later in pragmatic analysis of mathematics talk:

> It would be a great mistake to suppose that vague knowledge must be false. On the contrary, a vague belief has a much better chance of being true than a precise one, because there are more possible facts that would verify it. If I believe that so-and-so is tall, I am more likely to be right than if I believe that his height is between 6ft. 2in. and 6ft. 3in. (p. 91)

More recently, Tiegen (1990) has captured the same notion - which he calls the 'preciseness paradox' - in the following, similar, terms. Suppose two speakers, P and V, give similar information (the date of an historical event, say), but P is more precise than V (who is vague). One would then suppose P to be better informed than V in the field (history) in question. However, disregarding the knowledge level of the two speakers, V's statement is more likely to be true than P's, because the set of conditions (dates) which would make it true is much greater. The tendency is therefore to trust the person who is most likely to be wrong. Tiegen confirmed this prediction empirically.

The same point is made yet again by the Oxford philosopher of language, John Austin (best known to linguists as the originator of the theory of speech acts, which I describe in the next chapter):

> And isn't it surprising that precision should be paired off with incorrigibility, vagueness with impossibility of verification? After all we speak of people 'taking refuge' in vagueness - the more precise you are, in general the more likely you are to be wrong, whereas you stand a good chance of *not* being wrong if you make it vague enough. (1962a, p. 125)

In a discussion of vagueness in *Sense and Sensibilia* (1962a), Austin recognises that 'vague' covers a number of concepts. As he puts it:

> 'Vague' is itself vague. (p. 125).

Others have encountered the same dilemma.

> Vagueness is not easy to characterise or define. One reason for this difficulty is that there appear to be a number of different conceptions of vagueness, and it is not clear just what they have in common. (Burns, 1991, p. 3).

> If one looks more closely at this vagueness one soon discovers that the term itself is rather vague and ambiguous: the condition that it refers to is not a uniform feature [...] (Ullmann, 1962, p. 118)

Austin continues: a description of something, say a house, might be pronounced 'vague' on account of one or more of a number of features ("not necessarily defects", says Austin, "*that* depends on what is wanted"), including roughness, ambiguity, imprecision, generality and inaccuracy.

The words which signify these aspects of vagueness are not themselves precise or uncontentious. Channell (1994, pp. 34-8) distinguishes between vagueness and ambiguity, whilst pointing out that an utterance can be both. Her main point is that vagueness is much more significant factor than ambiguity in real communication, because ambiguity is usually automatically resolved by hearers, whereas vagueness "often plays an important part in the act of meaning".

Central to an understanding of vagueness in use, is Austin's comment (1962, p. 125) that vague features of language are:

> not necessarily defects, that depends on what is wanted.

His brief but important contribution to the discussion of vagueness affirms what Russell proposed in rather a grudging way: that vagueness in language can be, in some circumstances, not a flaw but a 'Good Thing'. Peirce, in defining "a proposition is vague when [...] it is intrinsically uncertain whether [the speaker] would have regarded [certain things] as excluded or allowed", is saying that vagueness offers speakers a way of saying something without needing to be sure of its scope of reference. From this perspective vagueness is not a limitation, but a means to *do* things which are inhibited by precise communication. Allowed vague components of language, I can still say *something* without having to say something precisely. But more than this, I can exploit the vagueness to convey something of my 'propositional attitude'; for example, I can let it be known that what I am saying is provisional.

MODALITY

Modality is a dimension of language which has some prominence in this thesis, because I am concerned with the ways in which speakers convey conviction, or the lack of it. From a semantic viewpoint, modality has to do with attitudes on the part of the speaker (or writer) towards the factual content of what s/he says.

Modal logic is traditionally concerned to distinguish between propositions that are necessarily true and those that are contingently true. Necessity and possibility are the two aspects of **alethic** modality. Propositions which are necessary truths are referred to as alethic necessities, those which are not necessarily false as alethic possibilities.

A second kind of modality - **epistemic** - is concerned, not so much with objective truth, as with human knowledge and belief. For example, the epistemic sense of "It may be raining" would be "I have reason to entertain the possibility that it is raining". The epistemic quality is explicit and clear in "I think that a bus will come soon". Epistemic modality enables the speaker to indicate her/his commitment to the truth of a proposition. Epistemic modals are included in "the general category of means used to convey the attitude of the speaker towards the utterance he makes" (Dubois, 1969, p. 118). An epistemic modal continuum has confidence and doubt at its extremes.

A third kind of modality - **deontic** - is related to the necessity and possibility of action. The appropriate notions here are obligation and permission. Thus: obligation = necessity to act, permission = possibility to act (Stephany, 1986, p. 376).

In English, modality is achieved syntactically in one of three possible ways:

- Mainly by the use of modal auxiliary verbs such as 'may', 'can', 'must', 'could'.
- By the use of epistemic adverbs such as 'possibly', 'maybe', 'perhaps'.
- Less commonly, by the use of verb moods and tenses. For example, "She appeared as though she were asleep" uses the 'modal preterite' i.e. past tense.

In a study of the modal auxiliaries 'must', 'should', 'ought', 'may', 'might', 'can', 'could', 'would', 'will', 'shall', Coates (1983) emphasises the importance of epistemic modality in normal (as opposed to logical) language (p. 18), and also the essentially subjective nature of epistemic modality (pp. 18-20), which serves to indicate the speaker's confidence (or doubt) in the truth of the proposition s/he expresses. Coates uses the term 'root' for some types of non-epistemic (including deontic) modality. This root category is problematic if one accepts Stubbs' position (1986, p. 15), that

> [...] all utterances express not only content, but also the speaker's attitude towards that content.

In effect, Stubbs declares that there is no root modality, that no utterance is neutral in respect to the speaker's commitment to what s/he is saying. Put differently, if all utterances were placed on a modal continuum from confidence at one extreme to doubt at the other, then there would be zero density in the middle. Yet it remains the case that the strength of the propositional attitude conveyed by an utterance is on a different, but related continuum. In particular, there is a strong case for believing that there is a root quality in mathematics - and that it is highly valued. Recall the words of Hardy:

> 317 is a prime, not because we think so [...] but *because it is so*, because mathematical reality is built that way. (1940, p. 130)

Hardy's passionate confidence is certainly epistemic, but when the commentary is stripped away from his uttererance, what remains is

> 317 is a prime,

which simply asserts what is the case. Surely many young men and women desert mathematics for literature because mathematics is *so numbingly root*. In their experience, not even their teachers seem to *care* much either way about its ideas and theorems; there is no passion in their experience of it, neither confidence nor doubt, no commitment of any kind.

In the end, I found a working distinction between root and epistemic to be helpful in relation to utterances of a mathematical kind. That is not to say that the distinction is clear-cut. I shall speak of epistemic modality in connection with commitment (or the lack of it) to hypothetical states of affairs, as opposed to perceived actualities, and particularly in connection with the attitude of the speaker to what s/he asserts about such contingent or hypothetical matters. Root modality will be (more or less following Coates) a more matter-of-fact kind of modality, not caught up in the belief or commitment of the speaker. I shall give evidence to show that epistemic modality is a prevalent and important means by which pupils mark tentativeness about their mathematical assertions.

PRAGMATICS AND VAGUENESS

As I have noted, Peirce's philosophy of pragmatism recognises the place of intention and interpretation in the determination of meaning. Pragmatics considers language from the point of view of the user - choices, constraints, purposes, and so on. To some extent, pragmatics has arisen in response to the limitations and artificial abstraction of truth-conditional semantics.

Pragmatics is a young linguistic discipline, and has suffered to some extent from a reputation as the 'waste-basket' of linguistics [Note 2.8] - to which to consign linguistic matters which were not the concern of 'pure' syntax or of truth-conditional semantics. As I shall show, the discipline of pragmatics is now richly endowed theoretically, and the 'waste-basket' reputation is manifestly *passé*. Both Levinson (1983) and Mey (1994) explore at length what it is that characterises pragmatics and how it complements syntax and semantics. Mey summarises:

> Pragmatics is the science of language seen in relation to its users. That is to say, not the science of language in its own right, or the science of language as seen and studied by the linguists [...] but the science of language as used by real, live people, for their own purposes and within their limitations and affordances. (1994, p. 5)

Now Lakoff's resolution of the problem of vagueness by means of (fuzzy) formal semantics attempts, by precise truth-valuation, to take the uncertainty out of vagueness - which is to ignore when and why, in the world, anyone should want to be vague in the first place. Or how communication can be possible, let alone effective, when (as Russell suggests) it is infused with vagueness. Such questions are outside the scope of truth-conditional semantics, but well within the province of pragmatics.

Lakoff's definition of 'hedges', "*words whose job is to make things fuzzier or less fuzzy*", conveys the presupposition that language can *do* things. Lakoff (1973, p. 490) and others (Prince *et al.*, 1982) include as hedges certain epistemic modal forms such as 'I think' and 'maybe' which hedge performatives, obscuring the degree of commitment of the speaker to what s/he is saying. Stubbs (1986) associates this with inexplicitness, "which implies vagueness and therefore deniability". *Thus, concern for meaning begins to entail a concern for speaker intention.*

In 1977, Sadock offered a radical pragmatic alternative to Lakoff's precise semantic truth-valuation of sentences containing `hedges. (Sadock more-or-less limited his attention to the word 'approximately'.) Sadock takes the view that:

> it is the *purpose* of the estimate that essentially determines how close to the truth it must be to be warranted. (1977, p. 434, emphasis added)

Later in the same paper he argues that:

> the role of an approximator [...] is *to trivialise the semantics* of a sentence, to make it almost unfalsifiable. (p. 437, emphasis added)

Extensive study of the purposes underlying vague language, including hedges, has been undertaken by Channell, who identifies (from empirical data) a number of goals which speakers and writers achieve by the use of vague expressions. These include:

- giving the right amount of information;
- deliberately withholding information;
- saying what you don't know how to say;
- covering for lack of specific information;
- acknowledging and achieving an informal atmosphere;
- expressing uncertainty;
- downgrading the importance of something so as to highlight something else;
- expressing politeness, especially deference;
- protecting oneself against making mistakes.

(Channell, 1985, 1990, 1994)

Many of these goals are evident in the mathematical conversations that I shall present and analyse later in this thesis. Channell's identification of these purposes presents an extremely useful pragmatic starting point from which to consider vagueness in such conversations. For example, it is not at all uncommon that novice speakers of mathematics have difficulty in giving expression to their mathematical thoughts, perhaps because they lack fluency in the mathematics register. In Chapter 4, I shall show some ways in which vague language enables such pupils to say what they don't know how to say. Similarly, uncertainty and lack of specific information are ever-present epistemic factors in any creative discussion of mathematics, where conjectures are asserted in advance of certain knowledge. The last of Channell's goals - protecting oneself against making mistakes - is associated with Sadock's notion of making a sentence "almost unfalsifiable". The use of hedges to introduce vagueness into propositions in mathematics talk will be examined in detail in Chapter 5.

SUMMARY

The world, insofar as it is entertained by thought or expressed in language, is infused with vagueness. This may be perceived as problematic if precision is, or is deemed to be, desirable, and is inhibited by vagueness. The problem of vagueness is only redeemable, if at all, by imposition of artificial syntax and interpretation, to achieve precision of expression and meaning in a stipulative way. Even vague language itself can be furnished with precise interpretation in such a way, recognising that this amounts to a formal 'game' of some kind, irrespective of relevance and truth. Mathematics itself, however, is seen to be fallible, shot through with uncertainty as to the origin and the truth of its propositions. The learning of mathematics may be viewed as a process of active construction in which, from time to time, vagueness is experienced individually and expressed socially. The enthymematic nature of inductive reasoning suggests that knowledge arrived at by inductive inference is provisional, plausible rather than certain.

This social perspective on learning presents an entirely different *pragmatic* perspective on the phenomenon of vagueness as a component of mathematical discourse. The thesis that I set out to demonstrate in subsequent chapters is that vagueness can be viewed and presented, not as a disabling feature of language, but as a subtle and versatile instrument which speakers can and do deploy to make mathematical assertions with as much precision, accuracy or as much confidence as they judge is warranted by the *circumstances* of their utterances as well as by their content.

My interest in vagueness *per se* arose because, as I shall demonstrate in subsequent chapters, vagueness turns out to be a unifying theme, common to many of the pragmatic aspects of language which I identified in mathematics talk in general, and in conjecturing talk in particular. As I have shown, a strict definition and circumscription of vagueness is problematic. It is useful to observe, however, that vagueness complicates the truth-conditional semantics of the propositional content of language, and that pragmatics deals with aspects of meaning that are outside the scope of truth-conditional semantics.

A principle which guides my choice of subject-matter is the aim to expose some of the ways that participants in mathematics talk use vague language for interactional and transactional purposes. My analysis occasionally extends beyond a strict focus on vagueness, in pursuit of insight into "the extremely subtle pragmatic interpretive

judgements regularly made by both teachers and pupils in the course of mathematics teaching and learning" (Pimm, 1994, p. 167).

In the next chapter, I review some linguistic topics. In particular, I survey some approaches to discourse which I shall refer to and apply later in the thesis.

CHAPTER 3: DISCOURSE AND INTERPRETATION

> Interpreting an utterance is ultimately a matter of guesswork, or (to use a more dignified term) hypothesis formation. (Leech, 1983, pp. 30-31)

> "I think your interpretation of this utterance might be controversial."
> (supervisor's written comment on student's essay)

OVERVIEW

In Chapter 0, I explained how and why I tape recorded and transcribed a number of mathematical conversations. These transcripts become the data which is analysed in Chapters 4 to 7. The purpose of the analysis was essentially threefold.

- To infer the transactional meanings of many of the utterances in these conversations, and in particular the way that individual children structure aspects of their mathematical understanding.
- To investigate how participants used and decoded language to support their contribution to a cooperative social interaction.
- To consider the motives which determine the character of that contribution to that mathematical interaction.

The pursuit of each of these purposes is an *interpretive* process. Whilst Leech's dictum (above) is superficially cynical, it would be optimistic to insist that it was entirely false. Nevertheless, I suggest that Leech's choice of the word 'guesswork' is not only undignified, but misleading. 'Guesswork' has an air of arbitrariness about it, so that it hardly matters whether or not the guess is wild of the mark, or a good guess, or remarkably accurate. The responsible analyst will take what steps s/he can to minimise guesswork by drawing on what is known about the process of analysis. In my own interpretive accounts of mathematics talk, I shall draw on the literature of linguistics - of pragmatics, in particular - with an eye to **regularities** and **purposes.**

- Where discourse linguists have found distinctive ways of describing and analysing aspects of language (in particular, in terms of indirect speech acts, implicatures and preference organisation) to be of interactive significance, I shall use these approaches to inform the analysis of my data. In this case, that data is overlaid with particular mathematical significance - the mathematical component of the context will be one that will also inform my interpretation.
- Where linguists have associated particular types of language (and for me,

Channell's work on vague language is the paradigm example) with particular goals, I shall look for those goals in situations where such language appears in mathematics talk.

The interpretation of the meanings and motives of others is, first and foremost, a synthetic act of meaning-making for the analyst, whose 'reading' of a particular utterance must be made to fit, to be consistent with, the way that s/he construes the utterance in its multi-dimensional context - social, psychological, mathematical, textual and who knows what else. The participants in the conversation are bound to be making such interpretations 'on the hoof' within the conversation itself, otherwise they could make no interactive contribution to it. The analyst after the event (in contrast with the clinical analyst-in-conversation) is subject to no interactive obligations, and so may 'interpret' at her/his leisure. At the same time, s/he has no interactive opportunity, and so is denied the possibility of testing interpretive conjectures (Steffe, 1991, p. 178) in order to validate or refute them.

I shall proceed now to outline a number of (mainly pragmatic) aspects of language, to which I shall refer in subsequent chapters. The survey is organised into two parts. The first of these examines the topic of reference, and pronouns in particular; the second considers some approaches to discourse. The topics which I shall address are:

REFERENCE

- pronouns, particularly the pronoun 'you'
- deixis

DISCOURSE ANALYSIS

- speech acts
- conversational implicature
- politeness theory
- conversation analysis

I shall show, at various points later in the thesis, how these issues relate to the analysis of mathematics talk. They will be seen to provide me with certain cues and expectations about the interactions, to inform my interpretation of the interlocutors' meanings and motives.

I begin with consideration of the bearing of language on the philosophical notion of reference.

REFERENCE

LANGUAGE AND REFERENCE

The theme of Chapter 4 is the value of vagueness as a means of referring to 'things' in mathematics talk. The linguist John Lyons, presenting a traditional view of reference (1968, p. 404), indicates that words refer to things. Thus, in the sentence "The book is blue", the definite noun phrase 'the book' is held to refer to some real-world object (a book). In a more recent exposition, Lyons (1977, p. 177) holds that it is the *speaker* who refers, by use of some expression. At heart, this is clearly a pragmatic, as opposed to lexical semantic, view of reference. In this view, reference can be viewed as an *action* on the part of the speaker/writer (Brown and Yule, 1983, p. 28). Successful reference is achieved by the speaker if the hearer (as interpreter of the utterance) is able to recover the intended referent(s). Such interpretation is bound to draw on aspects of the context of the utterance, especially in conversation, and speakers depend on this to achieve economy of utterance. In the utterance (father to son) "Have you fed the cat?" the son might be expected to pre-suppose that his father was referring to their household cat, rather than someone else's. When the son replies "Yes, I fed her this morning", the pronoun 'her' is then expected to be understood by the father to be 'anaphoric', that is co-referential with his earlier 'the cat'. Hence, the father would infer that, in using the pronoun, his son was referring to the household cat.

Some precision of reference is achieved by the use of names (e.g. 'Windsor Castle') and noun phrases (e.g. 'the man at the bus stop'). The referents of pronouns are *potentially* vague, in the sense of ambiguous or indeterminate, if they cannot easily be associated with a co-referential name or noun phrase. Context then becomes even more crucial for the purpose of interpretation.

PRONOUNS AND REFERENCE

The intended referent of a pronoun may be associated with a name or noun-phrase elsewhere in the text. Thus, in the sentence -

I:3.1 The next candidate to be interviewed was well-qualified for the job, and the panel agreed that **she** was impressive. [Note 3.1]

- both the pronoun 'she' and 'the next candidate to be interviewed' refer to the same person (i.e. the speaker uses both to refer to the same person). The use of the pronoun is anaphoric, being co-referential with a noun phrase which was uttered earlier in the discourse, in this case earlier in the same sentence. Of course, pronouns may refer forwards ('cataphora') as well as backwards, as in the demonstrative

1:3.2 **This** will be the last time I shall ask you to help me.

In the case of both anaphora and cataphora, the pronoun clearly relates to another item or items in the text or utterance; use of the pronoun is economical (Leech *et al.*, 1982, p. 191), and lends interest by avoiding the tedium of repetition of a noun or noun phrase.

It is not unusual for speakers to use pronouns in an irregular, somewhat anarchic way, with the pragmatic effect of conveying a range of social dimensions and attitudes to themselves and their audience:

> When **I** got there (Oxford), **I** think the first thing **I** learned was that for the first time in **my** life **you** were totally divorced from **your** background. **You** go there as an individual. So what did **we** learn? (Margaret Thatcher, ITV interview, 29 March 1983, quoted in Rees, 1983).

In this example, the speaker uses singular and plural, first and second person pronouns co-referentially. There is a fascinating shift from very personal recollection ('I') to description of shared experience conveyed in the second person ('you'), to reflection with hindsight ('we'). Knowledge of the speaker prompts the suspicion that the 'we' is a royal plural, a pronoun used by the (then) Prime Minister to refer to herself. The pronouns are used to code aspects of personal identity and group association.

REFERENTS OF 'YOU'

In Chapter 4, I shall demonstrate some of the subtle ways that children make use of particular pronouns in mathematics talk. The use of 'you' to refer to generalities is familiar in mundane language use, but has attracted little analysis in the literature. The exposition in this particular section therefore requires reference to evidence beyond my own data corpus in order to establish the perception that I shall apply to that corpus.

Helen Simons (1981, p. 39) records the following fragment of an interview with a 15-year-old girl. The subject was participation in class discussions.

HS:	Did you feel ... that you did have things to say?
P:	Yes. But often other people said them ...
HS:	And that put you off saying something another time did it?

P: Umm. If you say something you sometimes think that if you say something wrong people are going to think it is funny.

Simons uses 'you' to refer to the girl. The informal way that the pupil uses 'you' is utterly familiar. She refers, not to Simons, but (presumably) to herself and other pupils. Simons makes no comment on this - whilst it would look out of place in a formal text, it is a perfectly acceptable use of 'you' in speech, of the following kind:

> The pronoun of the second person may be used vaguely to denote some one (often the speaker himself) to whom something happens, or may happen, in the ordinary course of events:
>
> It was not a bad life. You got up at seven, had breakfast, went for a walk, and at nine o'clock you sat down to work. (Zandvoort, 1965, p. 128)

Now this kind of 'you' is vague in that it is unclear who is meant to be included by it, the only certainty being the speaker "himself" (by implication, since s/he does not choose a third person pronoun). This exemplifies the kind of reference indeterminacy associated with generality (Chapter 1). The fragment of dialogue which follows was broadcast on Radio 4 on 24th February 1995. Sue Lawley, host of *Desert Island Discs*, interviews Jimmy Knapp, President of the TUC:

Lawley: Are **you** an emotional man? Can **you** be moved by music?

Knapp: Aye, I think **you** can. There's a stirring in the breast that **you** can't deny.

Lawley's questions are clearly addressed directly and personally to Knapp. He is, after all, her chat show guest, and she wants to know what kind of a man he is. The potential ambiguity of her 'you' is demonstrated by his response, in which he deflects the spotlight from himself to comment - with uses of 'you' of the kind indicated by Zandvoort - on the general effect of music on human emotions. Knapp's 'you' is deliberately *impersonal*, and vague as to who is comprehended by it. This ambiguity was mischievously exploited in the *Clive James Show* (ITV 16th July 1995). Addressing an image of Margaret Thatcher on a giant video screen, Clive James, with a glint in his eye, suggests that there was once something between them, and asks the image

James: What do you suppose that did to me, an impressionable young man?

to which the image replies, in a recorded quotation, wickedly out of context:

Thatcher: I think it gave you a flavour of what life was all about.

My supposition is that Thatcher's 'you' was originally a reference to some shared experience, as used by her in the quotation about life at Oxford given earlier in this chapter. James exploits the ambiguity of her 'you' to suggest that she is addressing him. The type of marked pronoun alternation, which I noted earlier in Thatcher's reminiscences of Oxford, is deployed for effect in literature. In one of his novels, R. F. Delderfield gives this portrait of Evan Rhys-Jones, bank manager and landlord, described by his bank clerk and lodger, Charlie Pritchard.

> He had.. a gravity that **you** could mistake for dignity until **you** adjusted to the maddening deliberation of his movements. It was this characteristic that fascinated **me** on that first occasion, so that **I** found **myself** wondering how long it would take him to to select a stick of celery, bring it up to his chubby jaws and produce the soft, carefully modulated snap, in contrast to his wife's regular volleys from across the table. **You** had the feeling that if **you** asked him to pass the salt the meal would falter to an uncertain halt, so in the end **I** compromised, watching him but listening to his wife's coy exploration of **my** non-existent love-life. (Delderfield, 1969, p. 13, emphasis added)

This alternation between first and second person pronouns, 'I/me/my' and 'you', has the effect of distinguishing experiences and feelings from detached observation and generalised objective comment.

In effect, 'you' is being deployed in place of the more formal indefinite pronoun 'one' which might be regarded as somewhat affected in English speech. There is some indication that there is a corresponding trend in French. Laberge and Sankoff (1980, p. 271), in a fascinating and relevant sociological study of Montreal French, remark that "*Tu* and *vous* [...] are now locked in combat with *on* for indefinite champion, a title *on* thought it had locked up". The issue is that of *generality* in relation to what is being asserted.

> a detailed study of the contexts of use of indefinite *on* shows that *tu* and *vous* can be used in virtually all of them. Perhaps the most central element unifying these various contexts is the theme of generality or generalization. [...] It is important to note that the indefinite referent here is always vague as to the possible inclusion of speaker and hearer: *anybody* "means" just that - possibly you, possibly me, or anyone else in like circumstances. (*ibid.*, p. 275)

As I remarked, this matter seems to have escaped analytical attention with regard to English speech. I take it up again in the next chapter.

DEIXIS

The link between the use of pronouns and the pragmatic notion of deixis will be important in Chapter 4. Deixis is concerned with aspects of meaning that are inaccessible without the provision of context; for example, the use of a word or phrase whose referent is determined by the context of its utterance. Deictic features of speech and writing correspond to what philosophers call 'indexicals', which reveal attributes of place or person. Deictic forms such as 'you', 'now', 'here' are effectively context-dependent variables, or 'shifters' (Mey, 1994, p. 90). Divorced from the context of utterance, their meaning may be ambiguous or obscure. The authors of the 1980 ATM booklet *Language and Mathematics* refer implicitly to the importance of deixis for communication:

> Everyday conversation is easy to understand, its meanings are clear, because we speak in the context of everyday. When we are in a bus shelter, and a bus comes round the corner, the words 'here it is' have a clear meaning. (p. 17)

Similarly, as an example of temporal deixis, observe the difference in the intended immediacy of 'now' in the two utterances below, and how this difference is clarified by the context:

I:3.3 I'm going for lunch now. [context: workplace]

I:3.4 I suggest you begin the next chapter now. [context: supervisor to student]

The word 'deixis' is usefully related to its Greek root *deiknumi*, meaning 'to show' or 'to point'. From this same root, the more familiar noun 'paradigm' derives - meaning an example which acts as a pointer to a general type. The Greek word has another meaning, namely 'to prove'. A *diknumi* (*sic*) proof (Fauvel, 1987, p. 5) is one which is presented - typically by means of a diagram of some sort - in such a way that no explanation is necessary, for one can 'see' the result and the argument. For example, a suitable arrangement of pebbles in pairs 'demonstrates' that the sum of two odd numbers is even. The example displayed - the arrangement of a particular set of pebbles - is generic (in the sense of Mason and Pimm, 1984), it *points* to a more general truth. In the diknumi proof, a train of thought is shared yet unspoken.

Mühlhäusler and Harré (1990) emphasise the primacy, in their view, of the deictic role of pronouns.

> From developmental evidence we know that the ability to use pronouns in their deictic function predates the correct use of pronouns in their anaphoric

> function. [...] whilst anaphoric pronouns almost invariably contain deictic information, the reverse is not true. One is led by such observations to conclude that *the deictic function of pronouns is their primary function* ... (p. 58, my emphasis. See also Note 3.2)

Roger Wales, in a survey of developmental aspects of deixis (1986) observes that:

> [deictic expressions] serve as a meeting point for semantic, syntactic and pragmatic aspects of language. This is because they are, to use G. Stern's (1964) term, contingent expressions. By this is meant that, to interpret them, the interpreter needs not only context-independent semantic information but also information which is contingent on the actual (or construed) context. They are used to direct the hearer of a communication towards some object or event. (p. 401)

Deixis is concerned with ways in which language draws on and points to context.

SOME APPROACHES TO DISCOURSE

Over the last forty years or so, a number of approaches to the analysis of discourse have arisen and evolved. The roots of these analytical traditions have been in disciplines such as philosophy, sociology and anthropology, and their particular emphases and contributions vary accordingly. I now consider some of these ways of approaching discourse, each of which offers some insight into interactive dialogue. My purpose is not to suggest which of these frameworks is best or most appropriate, but (in the spirit of Schiffrin, 1994) to make a number of approaches available, because I draw on them in my own analysis in this thesis.

The first approach is based on Austin's insight that an utterance can be a means of performing an action.

SPEECH ACTS

An account of how speakers "do things with words" can be seen as an outcome of the theory of 'speech acts' which, for three decades, has occupied a central place in pragmatics. A declarative utterance such as "The window is open" expresses a proposition with a truth-semantic value - true or false. By contrast, the imperative utterance "Shut the window" does not express a proposition in the truth-conditional sense, since it cannot, under any conditions, be evaluated as true or false, or even something in-between. It is an order, requiring the hearer to do something; an action performed by language (speech in this case).

A quite different example of such an action is "Good luck!", which is a wish (*may you have ...*), a projection of a felicitous state of affairs in the near future. I can assert the truth of my sincerity in making the wish, but not the truth of the wish itself.

This insight is due to Austin (1962b), who called non-propositional requests, wishes and the like 'speech acts'. The essential property of speech acts is that they do something in the world, that they bring about (or have the potential to bring about) a change in some state of affairs. They are 'performative' utterances.

A speech act is accomplished, canonically, by the explicit use of a performative or speech act verb (SAV) such as 'promise'. The non-necessity of an explicit SAV in a speech act is born out by the "Good luck" example. The formal (if somewhat odd) paraphrase "I (hereby) wish you good luck" makes the SAV explicit.

Certain paradigm performatives powerfully convey the character of speech acts. When a priest says to the infant cradled in his or her arms "I baptise thee in the name of the Father, etc." then, as Mey (1993, p. 112) says "there will be one more Christian among the living". The standard bench test for a performative verb is whether the adverb 'hereby' can be sensibly, even if unnaturally, inserted into the utterance containing it. Thus, "I hereby request you to shut the window" stands up to the test, whereas "I hereby go to work by car" does not. Austin's initial position concerning speech acts was that they had to be associated with (possibly implicit) SAVs. He therefore urged the value of compiling a list of explicit performative verbs, a task that he judged would be "a matter of prolonged fieldwork" - by which he meant using the 'hereby' test to uncover the SAVs in a dictionary.

Force and Felicity

A speech act has three different kinds of 'forces' (Austin 1962b), as follows:

Locutionary force: the actual act of speaking.

Illocutionary force: the direct, conventional action of making a promise, request, command, denial, etc.

Perlocutionary force (or effect): the indirect (and sometimes unpredictable) consequences of the speech act which arise from circumstances of, and beyond, its utterance.

Example: "For tonight's homework I want you to finish the exercise". Illocutionary force - an order to finish the exercise before the next lesson. Perlocutionary effect - three pupils are very late to bed that evening.

In order to qualify as a speech act, the utterance in its context also has to satisfy certain 'felicity conditions' (Levinson, 1983, pp. 239-40), otherwise it does not 'count' as a properly-performed speech act . When the priest says to the child "I baptise thee", his or her pronouncement is felicitous provided that the sacrament is properly convened and the priest is invested with the authority to perform it. A more mundane example: if I ask you to close the door, the request is felicitous if the door is open, if I desire it to be closed, and if your situation is such that you are capable of closing it. On the other hand, a promise is not a performative if I have no intention of honouring it; nor is a million pound bequest, if I don't have that kind of money.

Implicit Performatives

Human rites, particularly religious and legal ones such as baptism by a priest or sentencing by a judge, are carefully formulated to include (if not deliberately) performative verbs which emphasise the fact that those who speak the words are doing something in the world. When the priest says, "I [hereby] pronounce you man and wife", this is the precise moment in which the speech act effects the union of two persons in marriage.

Consider an earlier statement, in the same ceremony, "I Jack, take thee Jill, [...] in sickness and in health". At one level, this is a description of something that is true at that moment, with no performative marking. But the importance of these 'solemnized' words is that they constitute a commitment (for example, to mutual fidelity) and a belief in a present and future state of affairs. Likewise, The verb 'to love' does not pass the 'hereby' test (? "I hereby love you") and so does not qualify, in Austin's strict sense, as a performative verb. Yet (I would argue) the declaration "I love you", simply cannot be uttered by the speaker as a pure and simple proposition, true or false: it has to convey an attitude, a commitment, a promise even, and in that sense it is a speech act.

Herein lies a difficulty for classical speech act theory, that a great many propositions can legitimately be viewed as 'acts' if the explicit performative requirement is relaxed. Austin himself recognised this difficulty and progressively adopted a broader view of speech acts. Levinson (1983) summarises thus:

> what starts off as a theory about some special and peculiar utterances - performatives - ends up as a general theory that pertains to all kinds of utterances. (p. 231)

Thus, the declaration (in the police station interview room) "My husband was at home in bed with me" can be viewed as a speech act, establishing an alibi, with the force of

"I hereby assert that my husband was at home in bed with me" (entailing that he was not simultaneously robbing a bank). This is the position of Austin's student and successor (as regards speech act theory) John Searle (1969), that every utterance can be classified as some kind of speech act: for example, the proposition "I am a man" is at the same time the speech act "I [hereby] assert that I am a man".

Indirect Speech Acts

The actual form of a speech act is not an arbitrary matter. Indeed, a whole range of personal factors - strategies, attitudes, beliefs, positions relative to other persons, desires, commitments and detachments - may be encoded in the way that a speech act is formulated. In particular, 'indirect speech acts' are "cases in which one illocutionary act is performed indirectly by way of performing another" (Searle, 1975, p. 60). Common forms of this are to state a preference or use of an interrogative form in order to convey a request. For example:

I:3.5 Teacher: I'd like to take in your exercise books.

I:3.6 Diner: Can you bring me the wine list?

These are both instances of how speakers frequently accomplish an indirect speech act by stating or questioning one of the felicity conditions (Gordon and Lakoff, 1971). In I:3.5, the teacher explicitly states his *wish* to receive the books i.e. that s/he meets the 'sincerity' condition (Levinson, *op. cit.*); in I:3.6, the diner questions the *ability* of the waiter to provide the list i.e. s/he questions one of the preparatory pre-conditions (*ibid.*). Note that I:3.6 is only conventionally interrogative, in that the waiter does not have the option of treating the question as a request for information rather than for action. I shall give particular attention to these matters as features of mathematics talk in Chapter 7, in the context of a Theory of Politeness. Traditionally, three major language functions are identified - statement, question and command - having typical realisations in declarative, interrogative and imperative verb forms. Indirect speech acts break down these canonical correspondences between language function and form. Thus, I:3.5 and I:3.6 achieve commands through declarative and interrogative forms respectively. Sinclair and Coulthard observe that:

> Modal verbs play a large part in producing the lack of direct correlation between the three grammatical forms and functions. (1975, p. 11)

Indirect speech acts are one of the means whereby a speaker may convey propositional attitude whilst at the same time making a declarative utterance. Consider an imagined scene from the mathematics classroom. The teacher singles out one child, and asks "How many lines of symmetry does a rectangle have?". The child might simply answer "Two". But it is possible to imagine circumstances in which the child may wish to convey that their answer is tentative, controversial, questionable or whatever. Some possible formulations include:

I think it's two.

Two, maybe.

Basically, it's two.

I'd say it was two.

My Dad said it was two.

In each of these cases, the illocutionary force of the utterance is hedged so as to convey doubt that 'two' is the correct answer, and to withhold full commitment to it. There is associated with each of them a 'hedged performative' (in some cases, tacit), so that what in a more confident speaker would be a statement of fact ("There are two") becomes an action signifying uncertainty. The use of hedged performatives in mathematics talk is a major theme of this thesis.

CONVERSATIONAL IMPLICATURE

I come now to the notion of *conversational implicature*, due to the philosopher Paul Grice (1975, 1989), who proposed that ordinary conversation is posited on a Cooperative Principle (CP), whose meaning is embodied in four sub-principles or 'maxims' of conversation. These maxims [Note 3.3] specify what participants need to do in order to converse rationally and cooperatively. The requirements are, essentially:

- maxim of Quality: let your contribution be truthful;
- maxim of Quantity: let your contribution be informative but not too informative;
- maxim of Manner: let your contribution be clearly expressed - e.g. be brief, orderly, unambiguous;
- maxim of Relevance: let your contribution be relevant to the matter in hand.

Now it is evidently *not* the case that all participants in all conversations observe all of these four maxims in all contributions. More often than not, this has nothing to do with an intention to lie or mislead (in which case the participation could not be deemed cooperative). Conversation can and does include utterances - such as refusals, disagreements and abuses - which would not be regarded as 'cooperative' in the ordinary meaning of the word. In what sense, then, does Grice use the word, given that participants will be expected (Grice, 1989, p. 26) to observe the CP? Grice himself does not attempt much of a gloss on his intended meaning:

> We might then formulate a rough general principle which participants will be expected (ceteris paribus) to observe, namely: Make your conversational contribution such as is required, at the stage at which it occurs, by the accepted purpose or direction of the talk exchange in which you are engaged. One might call this the Cooperative Principle. (*ibid.*)

One can proceed from this start in two ways. Mey adopts a position which holds that a contribution may *fail* to be cooperative. At least, this is the clear message of his telling and analysis of an anecdote in which a family 'friend' is abominably rude to his six-year-old daughter. Irritated when Sarah loses her ball in his book-lined study, he flaunts his erudition, to the bewilderment of the child, by directing her to "look behind Volume 6 of Dosteyevski's collected works". My human sympathy is with the father, who tersely comments (Mey, 1993, p. 67):

> the adult interlocutor failed to observe the principal demand set up by Grice in the CP: namely, to cooperate with your conversational partner.

On the face of things, the Dosteyevski utterance violates the maxims of Manner and of Quantity. There is, however, an alternative reading of the incident. Being cooperative is not the same thing as being pleasant. Since Grice is sparing in his commentary on the meaning of 'cooperative', one could suggest that the essence of cooperation is an intention to assist other participants in their pragmatic efforts to make human sense of the spoken interaction. Thus, in being angry or nasty, or in making a complaint, we may fail to be 'nice', but we certainly assist the communication of our meanings, feelings and attitudes. In fact, I suggest that Mey may be mistaken in his assumption that the friend's remark (which Mey takes to be uncooperative) is addressed to the *child*. Surely the intended audience was in fact the father, and the intended implicature something like "I'm irritated because your daughter is playing in my study", or perhaps "Please remove your daughter from my study".

How, then, do hearers interpret speech which is supposedly cooperative, yet which is for example, superficially untrue or irrelevant? Viewed from the other side of the coin, how do speakers successfully violate the maxims in order to communicate fine nuances of meaning, to enable the hearer to read 'between the lines' as it were?

The genius of Grice's theory is the following recognition. Whilst speakers do not always observe the maxims at the surface level, nevertheless hearers interpret the contributions of other participants in conversation as if they were intended to observe the maxims at some level of meaning other than that contained in the semantic content of the utterance. This has proved to be a robust theory, concise yet surprisingly complete, with a wide field of application, finding resonance with common sense and experience. For Grice describes and explains what we all know - that "communication involves the publication and recognition of intentions" (Sperber and Wilson, 1986a, p. 24).

Such a view of communication underpins a means of pragmatic inference identified by Grice, and which he named 'conversational implicature'. On many occasions, I shall show this means of inference in action in mathematical conversation. To clarify the meaning and process of implicature, consider the following exchange:

I:3.7 Teacher: Why haven't you brought your calculator to my lesson?

I:3.8 Pupil: My brother has a maths exam today.

The pupil's reply, taken literally, is irrelevant to the teacher's question. Indeed, the pupil appears to be flouting the maxim of Relevance. We could interpret the pupil's contribution to be simply non-cooperative, failing to address the teacher's enquiry about the missing calculator. In practice, we interpret the exchange as cooperative at some level, albeit not a superficial one. and so we infer, from the ostensibly irrelevant I:3.8 that:

I:3.9 my brother has my calculator, because he needs it for his exam.

The inference is an example of a (conversational) **implicature**, and we say that I:3.8 implicates the conclusion I:3.9. An implicature is rather like a hint. Thus, the human tendency is to accept Grice's theory as an accurate insight, one that exposes and codifies that which we 'knew', but had not yet isolated.

It is not my intention, here, to discuss the mechanics of pragmatic inference, i.e. how implicatures might be 'calculated'. Suffice to remark that the hearer brings his or her 'cognitive environment' (Sperber and Wilson, 1986a) to bear on the situation, for the purpose of interpretation. This 'environment' consists of a (very large) set of facts which are manifest (*ibid.*) to individuals by knowledge or by assumption (including, for example, the materials usually needed by candidates in a maths exam). What matters here is that pragmatic implicature is different from logical implication, in that the inferred conclusion - I:3.9 in the example - cannot be obtained solely from what has actually been uttered by application of a syllogism, or other process of logical deduction.

A useful way of looking at the Gricean interpretive framework would be to say that either speakers are overtly cooperative because they observe the maxims, or they are implicitly cooperative by setting up implicatures by means of maxim violations. In this technical sense, though not in the everyday sense, one can argue that *every* contribution to conversation is 'cooperative'. This is convenient in that Grice's maxims then become absolutes rather than mere ideals. Whenever possible, I shall approach the CP and the maxims with the expectation that speakers intend their interlocutors to make sense of what they say, even though they will not always (superficially) simply *say* what they mean.

For a final example, consider the hedged performative (promise, in fact)

I:3.10 Maybe I'll come and visit you next week.

which flouts the maxim of Manner, and thereby implicates

I:3.11 I may fail to come to see you next week.

because, if I were firm in my desire and intent to come, I would have made the promise without hedging the speech act. Hedges will be the linguistic focus of Chapter 5. I shall show how these words and phrases can be seen to act as indicators of propositional attitude in mathematics talk. Grice's theory of implicature accounts for their communicative effectiveness in conveying uncertainty, lack of commitment.

POLITENESS THEORY

Discussing a mathematical investigation with a 14-year-old boy, Allan, his teacher asks:

IRG5:51 Judith: Right. Can you make any predictions before you start?

The indirect form of Judith's request for a prediction (questioning a felicity condition, i.e. the boy's ability to provide it) is very characteristic of many of the teachers whom I studied through transcripts. In fact the boy made a prediction, but the vagueness of his answer suggests to me that it was far from secure:

54 Allan: The maximum will probably be, er, the least'll probably be 'bout fifteen.

(A full discussion of Judith's conversation with Allan will follow in Chapter 7.)

A particular quality of insight into such indirectness and vagueness in classroom mathematics talk is provided by a sociolinguistic theory - of 'politeness' - due to Penelope Brown and Stephen Levinson. First published in Goody (1978), the authors re-issued their account with minor revisions in Brown and Levinson (1987).

In essence, politeness theory is constructed to account for some indirect features of conversation; it claims that speakers avoid threats to the 'face' of those they address by various forms of vagueness, and thereby implicate their meanings rather than assert them directly. More often than not, Grice's principles explain how their intention is conveyed to hearers. An outline of the theory now follows.

Politeness theory is based on the notion that participants are endowed with two properties - rationality and 'face'. 'Face' (Goffman, 1967) consists of a public self-image, with two 'wants':

- 'positive face' - a desire to be appreciated and valued by others; desire for approval;
- negative face - concern for certain personal rights and freedoms, such as autonomy to choose actions, claims on territory, and so on; desire to be unimpeded.

The Model Person (MP) of the theory not only has these wants her/himself, but recognises that others have them too; moreover, s/he recognises that the satisfaction of her/his own face wants is, in part, achieved by the acknowledgement of those of others. Indeed, the nature of positive face wants is such that they can only be satisfied by the attitudes of others.

Now some acts ('face threatening acts', or FTAs) intrinsically threaten face. Orders and requests, for example, threaten negative face, whereas criticism and disagreement threaten positive face. The MP therefore must avoid such acts altogether (which may be impossible for a host of reasons, including concern for her/his own face) or find ways of performing them whilst mitigating their FTA effect, i.e. making them less of a threat.

Imagine, for example, that someone says something that MP believes to be factually incorrect. MP would like to correct them. Such an act would threaten the first speaker's positive face - the esteem in which s/he is held as a purveyor of knowledge. Or suppose that MP would like someone to open the window, but is aware of the threat to the other's negative face. Brown and Levinson identify a taxonomy of strategies available to MP in such circumstances.

1 *Don't do* the FTA - simply agree or keep quiet.

2 *Do* the FTA: in which case there is a further choice of strategy:-

2.1 Go *off record* - don't do the FTA directly, but implicate it e.g. "Don't you think it's hot in here?" (indirect request to open the window).

2.2 Go *on record*: either

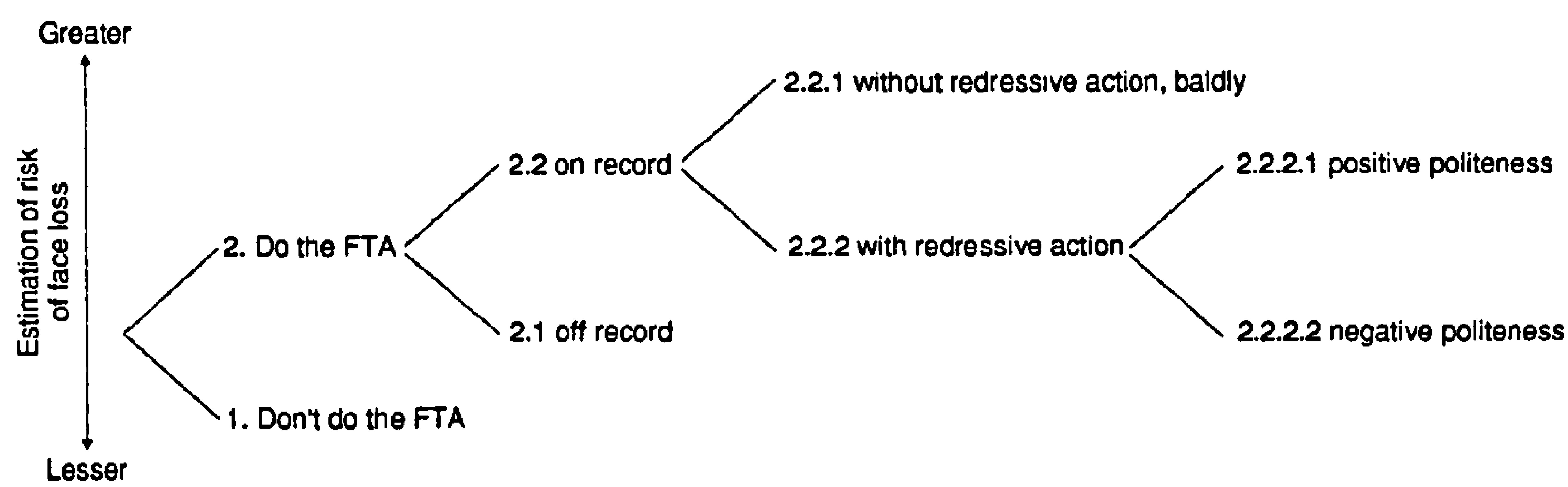
2.2.1 '*baldly*' - essentially making no attempt to respect face; or

2.2.2 with *redressive* action: having regard either for the other's

2.2.2.1 *positive* face ("You're the expert in these matters, but I thought that ..."; see also this chapter heading); or

2.2.2.2 *negative* face ("I'm sorry to trouble you, but would you mind ... ")

These strategies are summarised in the diagram below, which is due to Brown and Levinson, (1987, p. 69).



Going on record baldly suggests a greater concern for one's own face than for that of one's audience. I associate it with being 'assertive'; it sounds like bullying, but may be the only way that someone in a position of weakness can avoid being ignored or bullied themselves.

Redressive action is very commonplace when FTAs are in prospect; such action is a way of indicating that no face threat is intended. The strategies in the lower part of the diagram offer the least such threat, those in the upper part the greatest.

It is important to note that whereas an utterance (request, criticism, etc.) most obviously stands to threaten the face of the addressee (H), it may in fact threaten the face of the speaker (S). For example, making an excuse or accepting an offer may offend S's negative face (s/he feels obliged to justify, s/he is in debt to someone). A confession or emotional outburst offends S's positive face (s/he is seen as less worthy, less in control).

Sociological dimensions of 'face'

Brown and Levinson (1987, pp. 74-84) - building on work of Brown and Gilman (1970) on the social connotations of second-person pronoun use - isolate three factors in the assessment of the seriousness of an FTA. These factors are:

- the 'social distance' (D) of S and H;
- the relative 'power' (P) of S and H;
- the absolute ranking (R) of impositions in the culture.

D, P and R reflect the extent to which the actors (S and H) perceive these factors as mutual knowledge, as opposed to some rating of actual power, for example. A value for each of these three factors is assessed (notionally, on a scale of 1 to n), against the following descriptions.

D is a symmetric function D(S, H) which measures social proximity. Notions of class and educational achievement, for example, might be factors in the assessment of D. More fundamentally, D depends on frequency of interaction and mutual exchange of goods (material and non-material). A high value of D, reflecting close social proximity, is generally associated with mutual concern for the other's positive face - such as concern for the self-esteem of a colleague.

P is an asymmetric function P(H, S) which measures the relative power of H over S, in particular the degree to which H can impose his/her own wants at the expense of S's. The sources of P are either material (such as physical force) or metaphysical (by virtue of metaphysical powers ascribed to H). Usually, both will be relevant. A high value of P will generally be associated with deference - such as a pupil might be expected to show to a teacher, or a teacher to the headteacher.

R is a culturally and situationally-determined constant R_x, indicating the ranking of the imposition of the FTA x. Thus an employee's request for 'time off' on account of a hospital appointment might (in appropriate circumstances) be ranked lower than a similar request to attend a child's school play.

The dependence of R on context is clear; Brown and Levinson also argue for the context-dependence of D and P. For example, a high-rated social distance (in the context of the home town) between a parent and his child's teacher might significantly lessen if they found themselves in the same Mediterranean holiday hotel. In addition, Brown and Levinson argue for the independence of the three social factors, and thereby justify the simple summation of D, P and R in a given situation, in the assessment of an FTA. It will not be necessary, here, to give more detailed exposition of Brown and Levinson's development and application of this model. The sketch that I have given should be seen to be sufficient and relevant for the purposes of this thesis.

Indirectness

Brown and Levinson identify and catalogue a number of linguistic strategies associated with the face-respecting options shown in the diagram above. These include use of indirect speech acts, including quasi-interrogative forms such as:

I:3.12 You couldn't just pop out and get me a newspaper, could you?

which redress the threat to the addressee's negative face, their autonomy, respecting their right to refuse. However, given high values of D and P for example, there may be mutual recognition that refusal is not a real option. The following example is from one of a number of interviews with 10 and 11-year-old children, which I shall examine in detail in Chapter 5.

T3:17 Tim: OK, now let's think about two numbers that add up to twenty. Would you like to start off Caroline?

Both Caroline and I know that this is a command, that the indirectness (marked here by the avoidance of the imperative) is *conventional.* Caroline knows that she has no option but to "start off". I am nevertheless, sincere in my wish to be seen by these young students to be gentle, considerate and non-threatening.

I shall draw attention to the prevalence of this kind of indirectness in Chapter 7.

CONVERSATION ANALYSIS

Broadly speaking, the pragmatic theories that I draw on in this thesis derive from the philosophical tradition of language analysis, which is mainly associated with sentence structure and the logic of meaning. These latter semantic interests have most recently extended into what Mey (1993) calls 'micropragmatics', by which he means those aspects of language use which are analysed by reference to individual sentences or utterances, or at most pairs or short sequences of such units of text. A number of philosophical contributions to pragmatics at this level have been outlined in this chapter, notably the notions of speech act and implicature.

One characteristic of the philosophical approach to language is the status of invented (or intuitive) sentences as data, an approach somewhat alien to the late twentieth century education research tradition in which sociology and anthropology are so influential - though not, perhaps to a pure mathematician whose only 'data' are self-generated. For the philosopher, introspection and appeal to intuition are valid elements of method and sources of data. The following invented exchange is quoted from Grice's (1989, pp. 32) exposition of conversational implicature:

A: Smith doesn't seem to have a girlfriend these days.

B: He has been paying a lot of visits to New York lately.

No source (such as a corpus of conversation) is cited, and it is assumed that none exists, other than Grice himself. The argument (that B implicates that Smith has, or may have, a girlfriend in new York) is not dependent on the claim that this is a fragment of a 'real' conversation. Similarly, Brown and Levinson's account of off-record FTAs proceeds (1987, p. 69):

> So, for instance, if I say 'Damn, I'm out of cash, I forgot to go to the bank today', I may be intending to get you to lend me some cash, but I cannot be held to have committed myself to that intent [...]

Is the conclusion ('I cannot be held *etc*') any weaker because no tape recording exists of a spontaneous utterance "Damn, I'm out of cash *etc*"? Gazdar (1979, p. 11) is explicit:

> I shall assume ... that invented strings and certain intuitive judgements about them constitute legitimate data for linguistic research.

Philosophers such as Austin and Grice have usually based their arguments on consideration of such 'invented strings'. Nevertheless, such an approach has some limitations. The most obvious is that the audience for the argument must accept the plausibility of the invented text; in effect, to agree that the proposed sentence or sentences might well occur in some actual discourse.

Harvey Sacks himself laid the foundation for an overtly empirical approach to the analysis of discourse in a series of lectures from 1964-72. Garfinkel coined the term 'ethnomethodology' to capture the notion of 'ordinary' people studying naturally occurring speech data. With this perspective, Sacks developed the approach to discourse termed conversation analysis (CA). In fact, CA does not limit itself to mundane, 'ordinary' conversation, and the term 'talk-in-interaction' (Schegloff, 1987) has gained acceptance to indicate the scope of CA. Indeed, some of the most notable contributions of CA have been achieved in particular institutional settings, such as law courts (Atkinson and Drew, 1979) and medical interviews (Frankel, 1990).

CA came into being as a response to some perceived inadequacies of Searle's speech act theory - arguably one of the philosophical frameworks *most* sensitive to social organisation, and offering great promise in the illumination of particular institutional practices. The work of Labov and Fanshel (1977) attempted to apply speech act analysis in a study of psychotherapeutic interviews. Such interaction is typically indirect, and in viewing meaning as fundamentally emergent from sentence or utterance, Labov and Fanshel were obliged to propose a large set of context-sensitive 'translation rules' in order to move from the surface form of an utterance to its force as an action. One essential shortcoming of the speech act approach to the analysis of discourse exposed in this study is the tendency to look for the acts performed by small units of speech. As Drew and Heritage comment:

> There can, by now, be no serious doubt that sentences and utterances are designed and shaped to occur in particular sequential and social contexts and that their sense as actions derives, at least in part, from such contexts. (Drew and Heritage, 1992, p. 12).

A defining feature of CA is the approach to conversation in terms of extended sequences of utterances. Such sequences themselves contribute to a rich view of 'context', which construes utterances (and the social 'acts' they perform) as doubly contextual (Heritage, 1984). First, such utterances and actions are *context-shaped*, in that their production takes place in both a local configuration of speech (the *co-text*) and action which precedes them, and also a wider spatial, temporal and inter-personal environment which contains that configuration. Second, utterances and actions are *context-renewing*, forming the immediate context for some next action in a sequence. In this way, the CA analyst takes a dynamic approach to context, in contrast to the 'bucket' theory of context which prescribes and proposes a fixed framework of 'context' to account for certain features of a given discourse.

A speech act-based approach to discourse which takes account of the sequential organisation of action had in fact been developed by Sinclair and Coulthard and their collaborators in Birmingham. In their study of classroom discourse (Sinclair and Coulthard, 1975), a model of interaction was developed in terms of sets of acts, moves, exchanges and transactions. One such regularity associated with teacher questioning is the three-part Initiation-Response-Feedback (I-R-F) cycle, the prevalence and relevance of which to mathematics classrooms has been widely acknowledged since Sinclair and Coulthard first drew attention to it. In one example (due to Pimm, 1987, p. 27), a teacher Initiates a cycle by asking his class how a route map on the blackboard might be communicated to someone in the next room. A pupil gives a one-word Response "Coordinates". The teacher follows up with evaluative Feedback, "[...] That would be a very good way of doing it", and immediately Initiates the next cycle with another question, "What do you mean by coordinates?".

Such attempts to specify discourse text in terms of such 'rules', to develop a kind of grammar of interaction, are viewed with suspicion by CA proponents such as Drew and Heritage:

> In their preoccupation with the rules for discursive action within a context, the Birmingham group tended to ignore the task of analyzing how mutual understandings are achieved by the participants [...] This engendered a related failure to specify in their model how participants show their orientations to the particular institutional context in which they are interacting. For example [...] their analysis failed to disclose the ways in which successive elements of the I-R-F sequence *constitute* its instructional character. (1992, pp. 14-15)

Drew and Heritage appear to discount the value of the Birmingham research in *sensitising* the transcript analyst to the structure of the text s/he is examining, and the opportunity it presents for expert theorising about the relevance of the 'rules'. For example, Pimm (*op. cit.*) makes it his business *precisely* to illuminate, with reference to particular fragments of transcript data, how the I-R-F sequence exposes "how participants show their orientations to the particular institutional context in which they are interacting", (e.g. the teacher's desire for control, the pupil's sense of "exam" questions in the classroom), and conversely how these orientations can be seen to give rise to the I-R-F sequence. Consideration of these same orientations might suggest that the I-R-F sequence is not a regularity to be found in, for example, mundane dinner-party conversation.

CA rejects the premature construction of theories and any consequent deductive inferences from them. The CA approach is fundamentally *inductive*, although this is by no means a distinguishing feature of CA. In keeping with the interpretive research paradigm, there is constant dynamic interplay between data, theory and analysis (Hooper, 1989, p. 52). Data consist of tape recordings and *detailed* transcriptions of those recordings, coding pauses, intonation, sound stretches, interruptions and overlaps, dysfluencies and so on, using special symbols devised by Jefferson. This renders fluent reading of such transcripts difficult, but that is not the transcriber's prime objective.

A piece of CA: Adjacency Pairs and Preference Organisation

A major object of inductive attention in CA is the organisation of sequences of utterances, and in particular the sequential partition of conversation into 'turns' (in the sense that participants 'wait their turn'). This has led to the creation of a literature on the management of conversational turn-taking, and the related notions of directionality (projection from one participant to another) and intersubjectivity (so that next actions display a particular understanding of prior contributions).

These notions came under consideration in discussions of the concept of adjacency pairs (Schegloff and Sacks, 1973) and preference organisation. The notion of an adjacency pair arises in consideration of paired utterances. A pair is initiated by one speaker with a 'first part':

I:3.13 A: Can you come over for coffee tomorrow?

which is followed by the second part:

I:3.14 B: Yes, thank you, I'd love to.

or perhaps:

I:3.15 B: Well, er, let me see, I probably ought to go and get the lawn mower fixed in the morning, and my brother said he might come over sometime, so it's a bit tricky really.

Some types of second parts are routinely 'dispreferred' in response to first parts such as questions, offers, requests. (The term 'dispreferred' is intended in CA to describe an empirical regularity, and not to imply any state of mind in the speaker). I:3.15 is a more complex, hesitant response than I:3.14, which is a 'preferred' second part, in this case acceptance of the invitation. The complexity of the dispreferred second part is an observed linguistic (CA) regularity. The more elaborate structure of I:3.15 marks it as 'dispreferred' (non-acceptance).

Levinson (1983, p. 336) lists a correlation between different types of first parts and preferred/dispreferred second parts:

FIRST PART	request	offer/invite	assessment	question	blame
SECOND PART					
Preferred	accept	accept	agree	expected answer	denial
Dispreferred	refuse	refuse	disagree	unexpected	admission

Characteristics of dispreferred seconds include (Atkinson and Drew, 1979):

- *delays*: such as pauses before delivery;
- use of *prefaces*: particles such as 'well', token agreements (I'd like to, but ..);
- *accounts*: carefully formulated, over-elaborate explanations for the dispreferred act.

In Chapters 5 and 7, I shall draw attention to utterances in mathematical discourse which exemplify these characteristics. Here, for example, a teacher (Hazel) asks a 10-year-old pupil (Faye) a question which presupposes a certain belief. Faye's reply is delayed a prefaced, marking it as a dispreferred second pair part:

IRG3:33	Hazel:	... why do you think that for certain?
34	Faye:	Because ... well, I don't know for certain but I think ... 'cos the numbers that we've done are quite close to the first ...

The second part disagrees with an assessment presupposed by the question. It is also an unexpected answer to a question.

The complexity of my intuitive example I:3.15 above (consistent with the characteristics of dispreferredness) can alternatively be interpreted in terms of (a) violation of the maxim of Quantity, or (b) redress of threat to A's positive face. Indeed, the characteristics of dispreferred seconds listed above are all consistent with the avoidance of FTAs; much of politeness theory could be reconstructed within CA. More fundamentally, Levinson (1983, pp. 345-364) has devised an ingenious CA re-interpretation of the notion of indirect speech act as a particular type of 'pre-sequence', on the basis of this framework of turn-taking and preference.

OVERVIEW: APPROACHES TO DISCOURSE

Levinson *(op. cit.)* is probably responsible for presenting CA as an approach to discourse which is in opposition to less empirically-based approaches. His somewhat adversarial approach has since been taken up by supporters on both 'sides' (see, for example, Hopper, 1989, p. 60; Mey, 1993, pp. 48-49). This is unfortunate, because each approach offers different kinds of insights into interactive talk, and the wise analyst would do well not to be exclusive when interpreting transcripts.

In a recent book, Deborah Schiffrin (1994) sets out to reconcile many of the differences between alternative traditions in discourse analysis. She identifies six such approaches, including those associated with Speech Act Theory, with the Gricean cooperative framework, with Interactional Sociolinguistics (embracing Politeness Theory) and with Ethnomethodology. Schiffrin illustrates the different ways that these approaches illuminate sample texts. She argues that

> All [these approaches] attempt to answer some of the same questions, e.g. How do we organise language into units that are larger than sentences? How do we use language to convey information about the world, ourselves

> and our social relationships? (p. viii)

Schiffrin concludes that:

> [...] all the approaches to discourse view language as social interaction, and all are compatible with a functionalist rather than a formalist paradigm. (p. 415)

One of six unifying principles identified by Schiffrin (p. 416) asserts that analysis of discourse is empirical; analyses are predictive, they produce hypotheses. This is important, because it characterises the approach I took to the transcripts which I accumulated in this research.

Elsewhere, Brown and Yule (1983, p. 22) similarly claim that:

> The discourse analyst, with his 'ordinary language data' [...] may wish to discuss, not 'rules', but **regularities**, simply because his data constantly exemplifies non-categorical phenomena. (author's emphasis)

This interest in regularities is indicative of my own approach to the analysis of the transcripts of mathematics talk. But, for me, those regularities must have some inferential significance because, as a mathematics educator, I cannot be content merely to describe them. I *do* want to try to get "behind conversation", to make inferences and conjectures about "what is really going on" (Levinson, 1983, p. 287) - transactionally and interactionally - in mathematics talk.

My analytical approach is eclectic. My study of transcripts has been formative of more general analytical methods and in that sense (for *me*) highly inductive. I come to the transcripts without particular *linguistic* expectations. I describe what I find there, and then I set about studying that regularity, that phenomenon.

SUMMARY

In this chapter I have laid out the linguistic elements of the analytical framework that I shall apply to the analysis of many mathematical conversations. It is a highly eclectic framework, with the aim of insight into pragmatic aspects of talk in interaction. Speech is perceived as performing actions of various kinds, the effects of which on others may be indirect. Similarly, the intended meanings of utterances may be superficially obscure. Grice's theory of implicature asserts that cooperation (in the sense I have described) is nevertheless assured in conversation, and that this principle is central to the very possiblity of pragmatic interpretation of various kinds of vague and indirect contributions to conversation.

Whereas considerations of politeness can be argued to account for many vague and indirect features of conversation, these same features can be viewed (from a CA perspective) as an empirical regularity of interactive talk.

In the opening chapter, I explained my decision to reflect the chronological order of my work between 1991 and 1995 in the order of presentation in Chapters 4 to 6. In the first of these chapters, I give an account of how I first became aware of the significance of deixis in my data. I shall demonstrate the crucial communicative and meaning-making function of deictic language elements of mathematical discourse.

CHAPTER 4: POINTING WITH PRONOUNS

> What is implied in the proper use of pronouns? Do children recognise them early and integrate them in their own speech with ease and total comprehension? (Gattegno, 1981, p. 5)

In this chapter, I shall argue that the contribution of context to the interpretation of vague utterances enhances the ability of speakers (particularly novice speakers of mathematics) to refer to aspects of their own mathematical thinking, and thus assists their communication of such ideas. The argument centres on the use of pronouns in mathematics talk, and is based on two generative case studies. [Note 4.1]

THE INFORMANTS

The early stages of my data collection were based on extended, weakly-framed mathematical conversations with two children, Susie and Simon. My technique in contingent questioning was substantially trialled and developed in a series of interviews with Susie. I then undertook a somewhat smaller case-study applying the same weakly-framed interview method; the subject of these three interviews, which took place over a Christmas vacation, was Simon, aged 12¾ at the time. Being my son, he may have been a captive audience, but he was interested and cooperative. Like Susie, he is a quick thinker, and he articulates his thinking well. Paradoxically perhaps, being in close family relationship with me, Simon probably sees me as an 'authority' figure in mathematical matters more than Susie does; he is aware of my background and what I do when I'm at work - and sometimes when I'm not supposed to be at work (Rowland, 1992b).

If only on grounds of availability, Simon was an obvious choice as 'informant' (the term given by linguists to a source of linguistic data). Yet Simon plays only a supporting role in this thesis, whilst Susie has the lead. It was the conversations with Susie that jolted my research into life, and affirmed my conviction in the value of close study of individual children. As Stephen Brown says:

> One incident with one child, seen in all its richness, frequently has more to convey to us than a thousand replications of an experiment conducted with hundreds of children. (1981, p. 11)

Susie was nine years old at the time that we began our extended mathematical conversations. For me it was an important period - for the particular purpose of investigating Susie's mathematical thinking, and with the general aim of developing my own competence in contingent questioning. We talked together for over three hours in total, initially on four occasions over a period of five weeks [S1-4] and again some months later [S5]. The transcripts run to nearly eight thousand words. [Note 4.2]

I had spent most of the Spring term with her class at her school. Her teacher favours children working and learning together, but Susie was a child who did not seem to thrive in a co-operative group situation; she rarely seemed to be impressed by the ideas of her peers. Conversely, her proposals were usually ignored by them, partly (I think) because her insights were frequently inaccessible, or were elaborated at a length beyond the attention-span of her audience.

However, I was frequently fascinated by her contributions to teacher-managed class discussions when she showed that she was articulate and willing to expose her thinking to external scrutiny: for example, in a debate to determine the cost of one ruler, ten of which cost £3.50, Susie volunteered, "It's 35p, 'cos you cross off a nought". This was immediately followed by other estimates and proposals which suggested that very few of the children had listened to Susie's contribution, or had regarded it as being especially significant. My own instinctive reaction was that she had chosen an appropriate operation, but that she was 'merely' rehearsing a rule she had learned somewhere to execute the calculation. Perhaps with the same thought in mind, her teacher returned to Susie, and invited her to say more to the class about her method. "You cross off a nought," repeated Susie, continuing:

> "If you have ten, and you take away nine ones, you have just the one left ... it's because you take away a ninth ... no, nine-tenths. So there's one-tenth left."

I was, and am, fascinated by the fact that Susie 'explains' division by ten by talking about subtracting nine-tenths i.e. take away nine-tenths of ten (she seems to be saying) and you're left with one, just like crossing off the nought. Appendix 2 is principally an analysis of Susie's idiosyncratic approach to a particular class of division problems.

On this and on other occasions, Susie gave evidence of confidence, efficiency, and an unusual self-monitoring capability in her mathematical thinking. As it happens, her reading and writing of English were at that time significantly behind her mathematical, scientific, and indeed her artistic attainments.

She had another quality which was invaluable for my research purposes; that is a quiet but determined intellectual independence (not to put too fine a point on it, stubbornness), coupled with a direct kind of honesty which can manifest itself as rudeness, as measured by conventional social norms. Consequently, I never felt, in our conversations, that Susie was saying what she thought might please me in preference to what she believed. She frequently interrupted me when it suited her to do so. I initiated our conversations, but I didn't feel that I controlled them. To this extent they were weakly-framed as well as 'contingent': Susie seemed very happy to think on her feet, and I was compelled to do the same. Originally, I transcribed our conversations in search of data about her imagery in relation to number concepts and operations. This is particularly evident in the opening passages of transcript S4. What I got turned out to be generally disappointing in that respect. However, on close inspection of the text, I uncovered a significant linguistic phenomenon, her use of certain pronouns.

THE EXCLUSIVE 'WE': PRONOUNS AND MATHEMATICS EDUCATION

In Chapter 3 of *Speaking Mathematically*, David Pimm discusses the use of the pronoun 'we' in adult social practice, in particular in mathematics pedagogy. The following dialogue is from a classroom excerpt (Pimm, 1987, p. 65) involving a teacher and a ten-year-old pupil. The problem under discussion is 26 - 17.

> Teacher: What column's that? The tens column. Right. And what do **we** do there?
>
> Pupil: **We** cross that one out ... and then **we** put one there.
>
> Teacher: **We** take a ...
>
> Pupil: Er ... er ... er ...
>
> Teacher: **We** take a ... What do **we** take from the tens column? **We** take a ten, don't **we**?

It is a ritual intonement of a procedure (in this case, for subtraction by decomposition) which has been imposed on the audience, the child. This particular algorithm (decomposition) seems to be a particularly rich source of teacher 'we's. Hilary Shuard transcribed a similar conversation (given in full in Shuard, 1986) around the 'sum' 42 - 25, which includes

> Teacher: [...] Why can't you do it?
>
> Pupil: Two ones.
>
> Teacher: You've only got two ones haven't you? You haven't got enough. Do you remember, when **we** went through these sums last week, what **we** said you had to do, if you hadn't got enough?
>
> Pupil: ... errr ...
>
> Teacher: What did **we** say you had to do?

It is improbable that the child is included in the 'we' in phrases like "what we said you had to do". The phrase could be intended to imply "What I said in your presence". Mühlhäusler and Harré (1990, p. 129) propose that when academics use 'we' in exposition, it is in order to draw the listener into complicity. They point out that the addressee is thereby trapped in tacit agreement, and so prevented from voicing hostile opposition by the special (if ephemeral) relationship that has been artificially forged between expositor and audience. [Note 4.3] I believe that the teacher's 'we' has much of this quality. Pimm (1987, pp. 69-70) suggests that the teacher is often associating herself with some other (un-named) person or persons. He argues that the teacher, by using the plural pronominal form, is sometimes appealing to an un-named 'expert' community to provide authority for the imposition of a certain kind of classroom practice. Like the editorial 'we' (Wales, 1980, p. 27), the effect is to associate the speaker with a select and powerful group, from which the audience is clearly excluded. The result is to discourage and devalue any sense that the child might make of the situation, and to urge acquisition of the 'proper way' of doing such 'sums'. Such appeals to the support and authority of unspecified others is not, of course, peculiar to mathematics. Wills observes that:

> 'We' seems to have the greatest *imprecision of referent* of all English pronouns, and therefore is the most exploited for strategic ends (Wills,1977, p. 279; emphasis added).

Prime Minister John Major, in a speech (12th June 1991, reported in *The Times Educational Supplement*, 19th July 1991, p. 5) to the Centre for Policy Studies asserted that:

> We have been engaged in the struggle to resist insidious attacks on literature and history in our schools.

Given Wills' observation, it is interesting to speculate whether Major was associating himself with the right-wing, self-appointed CPS, or with the elected government.

I now turn to consider some ways in which Susie used two particular pronouns in our mathematical conversations.

'IT'

Initially, I was struck by the frequent appearance of the personal pronoun 'it' in our dialogue. By way of illustration, consider the following extract from our third session.

S3:68 Tim: What about this one you did; two hundred and sixty divided by ten is what?

69 Susie: Twenty-six.

[Tim writes 260÷10=26]

70 Tim: Right. And what's twenty-six times ten?

71 Susie: Twenty-six times ten ... twenty-six lots of ten ... ten lots of twenty-six ... oh, it's with forty it doesn't work. With forty I don't think it ... except with ones and tens and ... ones and tens. It wouldn't ... and twenties, sometimes twenties, ... em, sometimes thirties, sometimes forties, sometimes fifties, sometimes sixties, sometimes seventies, eighties, nineties, ...

72 Tim: You mean ...

73 Susie: If you do ten..and I think it would be ten if it, ... I don't, I'm not sure ... suppose you had 266 ... I'm not sure about this, I'll just find out.

74 Tim: Yes, you experiment and find out

[Susie writes 266÷10=26.6]

75 Susie: That's twenty-six point-six. And twenty-six point-six lots of ten ... so you'd in a way put a nought on the end, but you'd end up like that [she has written 26.6x10=266]

76 Tim: When you say ... [interrupted]

77 Susie: So any tens with any other, with any number, it would end up like that.

78 Tim: With tens ... [interrupted]

79 Susie: [forte] and the same with ones, but not with something like sevens, or whatever. And sometimes with twenties and thirties and forties and fifties and sixties and so on.

80 Tim: What's another number like seven that you think it wouldn't work with?

81 Susie: It wouldn't work with ... [writes 30÷7=]

82 Tim: You're doing seven again?

83 Susie: Yep, I thought you asked me for seven.

84 Tim: I said, it doesn't work with seven, is there another number like seven that it wouldn't work with?

85 Susie: Oh, with forty, keeping the forty?

86 Tim: I don't mind, you can change it if you want. I mean, is it the number your dividing by that makes it work, or the number you're dividing into ...

87 Susie: I'm not sure if five does do it or doesn't do it. So could I find out if five does do it?

88 Tim: Of course you can.

89 Susie: But I'm not saying that this one will not work, OK? [writes 30+5] I want to know whether it's doing it or not. [pause] Six. And then six lots of five; five lots of six [writes 6x5=30] . It does work.

90 Tim: So it works with five.

91 Susie: With tens ... ones, fives and tens it probably always works. And sometimes fifteens, twenties, twenty-fives, [then very fast] thirties, thirty-fives, forties, forty-fives, fifties, fifty-fives, sixties, sixty-fives, seventies, seventy-fives, eighties, eighty-fives, nineties.

92 Tim: It works with all those numbers d'you think?

93 Susie: Sometimes it works. Sometimes.

I selected the passage above in order to make a point; but is it misleading? I shall consider the full corpus. But in any case, how to judge whether the occurrence [Note 4.4] of the neuter third person singular pronoun in that corpus is in any way unexpected, or untypical? To make a start on this question, a list of the words which children use frequently is needed, such as that compiled by Rinsland (1945) in the USA, from children's writing and conversation. More recent studies of the vocabulary of English children include those of Burroughs (1957) from speech data, and Edwards & Gibbon (1973) from children's spontaneous writing. [Note 4.5] Despite their differing methodologies, there is a high level of agreement between these two studies about the ranking of very common words. The speech/writing data for both of these studies were from children up to two years younger than Susie. However at age 7+ the nine words used most frequently are (in decreasing order of popularity):

and, the, a, I, to, it, is, my, go

In fact the first five are way out in front on the basis of a 'popularity index' [Note 4.6] used by Edwards and Gibbon, followed by 'it' and the other three above, which are about equal to each other.

So I compared the incidence of 'it' with that of these other eight words in my transcripts, for each of my meetings with Susie, and aggregated over the full 7760-word corpus. My findings on these eight words are shown opposite in Table 4.1. Table 4.2 overleaf gives additional comparative data on the first eight of these words and others which occurred most frequently in the transcripts.

The frequency-trend for the first five words in Table 4.1 is much as expected from the studies of Burroughs, Edwards and Gibbon, and indeed from broadly-comparable studies (in particular that of Howes, 1966) of adult language. [Note 4.7] The seventh-ranking 'is' (according to Edwards and Gibbon) occurs significantly less frequently (less than half as often) than those first five, as expected; in fact, it is ranked 12th in the Susie corpus. I find it unsurprising that appearances of 'my' and 'go' are rare in our maths talk. The observation that Susie makes extensive use of 'I' in the first session may be accounted for by the fact that she was at that time acquainting me with some of her personal approaches to arithmetic; perhaps Susie also felt the need to assert herself most in our initial intellectual manoeuvres.

	S1	S2	S3	S4	S5	ALL	PER 1000
and	41	44	47	42	55	228	29
the	40	71	39	26	51	227	29
a	29	48	26	28	89	220	28
I	47	38	16	25	63	189	24
to	28	47	11	30	41	157	20
it	**26**	**57**	**48**	**28**	**64**	**223**	**29**
it's	**13**	**10**	**11**	**2**	**13**	**49**	**6**
is	17	15	20	6	32	90	12
my	0	0	0	0	0	0	0
go	0	0	1	4	4	9	1
LENGTH	1130	1810	1310	1060	2450	7760	1000

Table 4.1

The pronoun 'it' (ranked sixth by Edwards and Gibbon) is consistently used more often in my conversation with Susie than these quantitative studies of natural language would lead one to expect, clearly ranking alongside 'and', 'the' and 'a'. Table 4.2 shows that effectively only 'you' (which I will consider later) occurs more frequently, and even that is no longer the case if occurrences of 'it's' are included. Indeed it (by which I mean 'it') is one of the lexical hallmarks of our conversation.

Pimm (1987, p. 22) remarks that:

> Like much informal talk, spontaneous discourse about mathematics is full of half-finished and vague utterances.

He proceeds to illustrate the point by drawing attention to the use of 'it' in an exchange (about enlargement) between a teacher and secondary pupil.

rank	word	S1	S2	S3	S4	S5	All	S1	S2	S3	S4	S5	All	WCU7+ [1]	Howes [2]	Howes	LOB [4]	LOB
		frequency						/1000	/1000	/1000	/1000	/1000	/1000	rank	/1000	rank [3]	/1000	rank
1	you	26	76	55	26	80	263	23	42	42	25	33	34	53	19	6	4	32
2	and	41	44	47	42	54	228	36	24	36	40	22	29	1	38	3	28	3
3	the	40	71	39	26	51	227	35	39	30	25	21	29	2	41	1	68	1
4	it	26	57	48	28	64	223	23	31	37	26	26	29	6	17	8	10	10
5	a	29	48	26	28	89	220	26	27	20	26	36	28	3	24	5	23	5
6	that	32	53	29	17	61	192	28	29	22	16	25	25	68	15	10	11	7
7	I	47	38	16	25	63	189	42	21	12	24	26	24	4	40	2	7	17
8	to	28	47	11	30	41	157	25	26	8	28	17	20	5	25	4	27	4
9	of	18	21	26	22	46	133	16	12	20	21	19	17	19	18	7	36	2
10	one	12	22	13	8	51	106	11	12	10	8	21	14	23	5	18	3	38
11	so	17	20	18	3	43	101	15	11	14	3	18	13	62	9	11	2	46
12	is	17	15	20	6	32	90	15	8	15	6	13	12	7	7	13	11	8
13	what	12	12	14	17	26	81	11	7	11	16	11	10	143	4	21	2	55
14	with	6	23	19	10	19	77	5	13	15	9	8	10	14	5	19	7	14
15	be	12	16	5	9	31	73	11	9	4	8	13	9	90	5	17	7	15
16	do	7	12	15	12	26	72	6	7	11	11	11	9	104	4	20	1.5	73
17	in	6	24	7	10	23	70	5	13	5	9	9	9	11	17	9	21	6
18	number	6	15	16	8	18	63	5	8	12	8	7	8	630	0	25	0.5	189
19	would	13	15	8	8	15	59	12	8	6	8	6	8	168	2	23	2.5	43
20	yes	4	13	6	8	28	59	4	7	5	8	11	8	183	1	24	0	322
21	like	4	29	9	2	9	53	4	16	7	2	4	7	61	5	15	1	83
22	now	7	15	6	5	18	51	6	8	5	5	7	7	172	3	22	1.5	72
23	it's	13	10	11	2	13	49	12	6	8	2	5	6	492	7	12	0.5	196
24	this	1	9	11	12	16	49	1	5	8	11	7	6	40	5	16	5	22
25	on	11	22	10	0	5	48	10	12	8	0	2	6	12	7	14	7	16
	LENGTH	1130	1810	1310	1060	2450	7760						1000					

Table 4.2 [1] Edwards and Gibbon (1973) [2] Howes (1966) [3] Rank within this set of 25 words [4] Hofland and Johannson (1982)

Teacher: [...] Its width is only a third as long as that one so - [...] how many of the smaller squares can you fill in? - nine, right.

Pupil: Is it that you square it - every time?

Pimm comments on the ambiguity of the referents for the occurrences of 'it', particularly in the pupil's question. He suggests that in saying "you square it", the pupil is making a generalisation. He picks up the "crucial expression ... *every time*" in evidence. I shall examine the pupil's use of 'you' to support his conclusion further. Incidentally, the pupil's first 'it' could well be seen as an embryonic form of the cataphora "Is it the case that ...", drawing on the philosophical register.

Building on Pimm's observation, and my awareness of the frequent occurrence of 'it' in my Susie corpus, I have considered the varied purposes for which 'it' is being deployed in the corpus, and suggest that 'it' is a distinctive and important feature of maths talk, to the extent that it acts as a linguistic pointer, invariant at the surface level. To investigate this, I considered (for each occurrence) what 'it' is referring to. What is it being used to mean? The variable character of the referent of 'it' is illustrated in the examples which follow.

Our first conversation began as follows:

S1:1 Tim: Imagine a square standing on its edge, on a table. Sketch what you see. [Susie draws the table and the square in 3D, showing edge of card]

2 Susie: It's the width of it.

3 Tim: Imagine a very thin square. The square rocks around on its bottom corners, jumps up, floats off the table, spins around, slows down, and drops back onto the table.

4 Susie: It's all floppy over ... it bounces and sort of flops down.

From the outset, in Susie's very first 'turns' [S1:2, 4] in our conversation, the 'it's are in evidence. There is a superficial and coincidental resemblance to Pimm's

Pupil: Is it that you square it - every time?

Squares feature in both situations, though one is algebraic (a product), the other geometric. The second 'it' in S1:2 and those in S1:4 are anaphoric as she makes reference to a square which I have already introduced in S1:1. Rather, she is referring to to her mental image of a square. There are questions that I could pursue about this geometric image (where is it?; what is it like?; can I get access to it?; of what class of squares is it a representative?). Note that "the table" is a viable (if unlikely) alternative referent in S1:4, on the basis of the interchange above; at the time I did not even consider that possibility, since my own attention was on the square - *my* square. With her first 'it' [S1:2] she refers to her drawing of the edge of the card; I am able to infer this from her gesture to the relevant part of the drawing. The pronoun is not co-referential with anything that has been (or is about to be) said.

This is an example of the linguistic phenomenon of **deixis**, by which a referent must be inferred by consideration of spatial, temporal, personal or other aspects of the *context* of speech (Levinson. 1983, Chapter 3). Thus, (above) the pupil's first 'it' is cataphoric, formulating the question that follows, but the second is deictic (referring to a ratio of lengths).

Towards the end of the second session, I say:

S2:94	Tim:	One last thing. You remember last week we were doing multiplying by five, and you said that's easy. What you did, you multiplied by ten, you added a nought, ...
95	Susie:	... and then you halved it.

Here the referent is a number, ostensibly the original number multiplied by ten. Or is it the original number with a zero tacked on the end (which may be the same thing in form only; but recall my tale of Susie and the Problem of the Ten Rulers)? Or does economy suggest that it is precisely the original number which Susie halves, leaving the zero to hold the end place? Like the teacher's 'we', Susie's use of the pronoun 'you' - referring to herself but perhaps not only to herself - is an interesting instance of participant deixis (Wills 1977); the 'you' has an indefinite, impersonal quality, to be considered later in this chapter.

The dialogue continues :

S2:98	Tim:	Now why, do you think, when you multiply by ten, you just add a nought.
99	Susie:	Because ...10 lots of ... if you count up in twos, suppose, the tenth will be twenty. And fours forty, fives fifty, and so on. Sixes sixty, sevens seventy, eights eighty, nines ninety, tens one hundred. And so on. ten lots of, that's just a nought on the end.
100	Tim:	That's extraordinary isn't it. Does that surprise you?
101	Susie:	Not to me.
102	Tim:	Why does that happen with ten though, and not twelve, or nine, or..Why does multiplying by ten add a nought on the end?
103	Susie:	Twenty would add two noughts, for instance.

104 Tim: So if I multiplied, say, three by twenty, that would give three with two noughts?

105 Susie: No. No, no, no, no, no. Silly me. Silly me. No. It's only with ten. For twenty you would double the number at the beginning.

106 Tim: Why is ten the magic number like that?

107 Susie: I don't ... it's just mathematics.

108 Tim: [laughs]. You mean the magic number might have been thirteen, or seven, but it just happens to be ten.

109 Susie: By how the mathematics is made. How it was invented.

110 Tim: Who invented it then?

111 Susie: No, I didn't invent it!

112 Tim: No, *who* invented it?

113 Susie: Don't know. They invented the mathematics and then people sort of added to it. It grew over the years.

114 Tim: Why did the people that invented the mathematics make ten the magic number?

115 Susie: I don't know [laughs]

In this case (S2:105 "It's only with ten") the referent is a symbolic procedure. She is noting a property which belongs to powers of ten, but not to all multiples of ten. The property in question is clear from the context above. Indeed I explicitly state the simplest version [S2:102]; Susie formulates and subsequently withdraws a generalisation of it.

I have included the concluding lines [S2:106-115] in which [109, 113] Susie makes anaphoric reference to 'mathematics', which she seems happy to discuss as an objective entity, even to the extent of giving it the definite article. Is it fanciful to suggest that Susie takes a non-platonist position *vis-à-vis* mathematics? She certainly sees it as something "invented" rather than 'just there' or discovered. The clearest statement of her position is given quite spontaneously:

S1:113 Susie: Don't know.They invented the mathematics and then people sort of added to it. It grew over the years.

I come now to some examples which show how Susie makes effective use of the pronoun 'it' to point to ideas of a general nature which neither she nor I have named, and whose nature I must be expected to infer.

Deixis

I shall demonstrate that Susie sometimes employs 'it' as a conceptual deictic i.e. to point to concepts. The following example comes from our third session.

S3:17	Susie:	No, no ... but times can do it can't it, and add, and take ... no, takeaways can't do it.

The second 'it' seems to have the earlier 'times' (multiplication) as referent. The first (and third) 'it' is more problematic. On the basis of these 18 words alone one can certainly surmise what the first/third referent - object of the verb 'to do' - might be. For example (given that Susie is nine) "which operations make bigger". Her syntax indicates that it is the operations themselves which can or can't do whatever-'it'-is. A more extensive context is required. A short lead-in gives some help:

S3:12	Tim:	Why is it that twelve divided by two is equal to six, then?
13	Susie:	Well what it is, is this number [12] and see how many times that [2] goes into there [12] . How many times two goes into twelve.
14	Tim:	ahh..two goes into twelve..
15	Susie:	or twelve goes into two
16	Tim:	or twelve goes into two
17	Susie:	No, no ... but times can do it can't it, and add, and take ... no, takeaways can't do it.

I am quite sure that Susie has introduced the concept of commutativity into our dialogue. Not only does she not name the concept, but she is probably *unable* to give a name to it; perhaps it would be very surprising if she could. However she certainly knows when 'it' holds. With the deictic 'it', used here as a provisional object (Zandvoort, 1965, p.135), she can articulate aspects of what she knows, and she does so quite spontaneously and unexpectedly. [Note 4.8]

I regard her use of deixis in my final example (which is a prelude to the extended extract from transcript S3 given earlier in this chapter) as even more fascinating. To set the scene: Susie had divided 56 by ten and written 5.6. She explained what point-six signifies:

S3:28	Susie:	It means it's six of the number you're doing it by, a tenth. Six-tenths.
29	Tim:	Ah, six-tenths.

30 Susie: It's six-tenths of the real number.

[Note 4.9]

I'm suspicious that she is using decimals as remainders. To test this hypothesis, I ask her for 56 divided by 17. The question is a contingent one, chosen with the aim of precipitating cognitive conflict; it is chosen to give 5 as remainder. Susie duly writes 3.5. Closing in, I ask her:

S3:37 Tim: What does that point five mean?

38 Susie: It's a half of a ... it's a half, of the real number. That's three of the real number and the point five means it's going to be tenths. The three is three whole numbers.

It now seems that her remainder-is-decimal rule is getting some interference from her confident knowledge that point five is a half. So next I set up 40 divided by 7. Susie writes 5.5, and again confirms that the point five is a half.

S3:40 Tim: Right, let's get this absolutely clear. Is that five whole ones and a half of one?

41 Susie: Yep.

42 Tim: So how many lots of seven in forty, it's five and a half?

43 Susie: Yep.

She continued, to my surprise, by volunteering an arithmetic deductive inference:

43 Susie: Yep. So seven lots of five and a half is forty.

Unawares, she has opened up a chink that I can exploit to 'correct' her generalised mis-construction about decimals. So I press home the cognitive conflict, and ask her to work out seven lots of five and a half [S3: 44-53]. She obtains 38.5, preferring to work with 0.5 for a half.

S3:51 Susie: You could do this [she writes 0.5÷7=3.5, corrects '÷' to '×']

52 Tim: So what's seven lots of five point-five altogether?

53 Susie: Thirty-five point-five ... thirty-six point-six ... no, thirty-eight point-five [writes 5.5×7=38.5]

Thinking that I have a checkmate situation, so to speak, I home in:

S3:60 Tim: Now what does that mean? [indicates 40÷7=3.5]

61 Susie: How many times does seven go into forty

62 Tim: When you actually work it out, five point-five times seven, you don't get the number you started with.

63 Susie: I know

64 Tim: Isn't that a bit funny?

65 Susie: No, that isn't, because whatever number you put in there [indicates 7] you'd never reach forty, except for one. And you're not allowed one

66 Tim: Why not?

67 Susie: 'cos it's not really in the maths, it's just one, two, three, four, five, six, seven, eight,.. [*meaning the one-times table?*]

Susie is unperturbed. When 40 is divided by 7, and the quotient is then multiplied by 7, she has no problem in living with a product which differs from 40, or so it would appear. In fact, Susie subsequently enters into a lengthy and self-driven exploration into what she considers to be 'special cases' in which divisor × quotient = dividend. First she considers 10, in which she has some confidence, but which she tests with a 'crucial experiment' (Balacheff, 1988, p. 218), taking 266 for the 'crucial' dividend.

S3:73 Susie: If you do ten..and I think it would be ten if it, ... I don't, I'm not sure ... suppose you had 266 ... I'm not sure about this, I'll just find out.

7 Tim: Yes, you experiment and find out

[Susie writes 266÷10=26.6]

75 Susie: That's twenty-six point-six. And twenty-six point-six lots of ten ... so you'd in a way put a nought on the end, but you'd end up like that [she has written 26.6x10=266]

Susie proceeds to consider whether 5 is also a 'special case', although her test case, involving 30, seems rather lenient in comparison:

S3:87 Susie: I'm not sure if five does do it or doesn't do it. So could I find out if five does do it?

88 Tim: Of course you can.

89 Susie: But I'm not saying that this one will not work, OK? [writes 30÷5] I want to know whether it's doing it or not. [pause] Six. And then six lots of five; five lots of six [writes 6x5=30] . It does work.

She goes on to make conjectures, and she articulates generalisations freely:

S3:90 Tim: So it works with five?

91 Susie: With tens ... ones, fives and tens it probably always works. And sometimes fifteens, twenties, twenty-fives, [then very fast] thirties, thirty-fives, forties, forty-fives, fifties, fifty-fives, sixties, sixty-fives, seventies, seventy-fives, eighties, eighty-fives, nineties.

92 Tim: It works with all those numbers d'you think?

93 Susie: Sometimes it works. Sometimes.

One week later, at our next meeting, Susie is able to recall, with remarkable fidelity, what it was that she had come, in a tentative way, to believe:

S4:31 Tim: OK, here's something left over from last week. We had $260 \div 10 = 26$ and $26 \times 10 = 260$. We also had $40 \div 7 = 5.5$ and $5.5 \times 7 = 38.5$ [Tim writes all these] OK? You remember that we talked about that?

32 Susie: Yes, but if five or ten you do it with, it always comes out the same number.

33 Tim: Yes, I was going to say that you said to me that sometimes ... [interrupted]

34 Susie: And sometimes fifteen, twenty, twenty-five, thirty, thirty-five, fifty, fifty-five, sixty, sixty-five, seventy, seventy-five, eighty, eighty-five, ninety.

35 Tim: Ninety-five?

36 Susie: Yes.

37 Tim: A hundred?

38 Susie: Sometimes.

39 Tim: A hundred and five?

40 Susie: Sometimes. Any by five or by ten will sometimes do it.

41 Tim: What will it do?

42 Susie: You start with the same number as you end.

Susie has abstracted a connection between dividing and multiplying which becomes the focus of so much of our subsequent conversation. She has no name for this relation, so she makes deictic use of the neuter third person pronoun, and frequently by saying that such-and-such a number (the divisor) will "do it", or in the phrase "it works" [S3:87, 89]. Her vagueness achieves for her the goal of covering for lexical gaps. Channell (1985, pp. 12-15) noted the same ability in her linguistics students in a tutorial, "to get across a meaning where they do not have at their disposal the necessary words or expressions which they need to associate with the concepts they are forming" (p. 12).

Recall that Pimm notes the significance of the "crucial expression ... *every time*" in the formulation of his pupil's generalisation. His (the pupil's) generalisation is offered in the form of a question, giving it a certain tentative rather than assertive quality. Likewise Susie knows that there is a generalisation waiting to be articulated, but she is uncertain about how comprehensive it can be. For convenience, denote by s(b) the sentence: "For all n, $b\times(n\div b)=n$." Now Susie is confident that s(5) and s(10) hold - this is evident in S4:32. Here, for Susie, the "crucial" word is 'always', used with identical meaning and effect as 'every time'. In no way is S4:32 tentative. On the other hand she suspects the truth of s(15), s(20) and so on but conveys her doubt in the word 'sometimes' which she uses repeatedly, and in contrast to 'always'. 'Sometimes' does not promise 'every time'.

This episode illustrates how *Susie's use of the deictic 'it' enables us to share and discuss a concept which Susie possesses as a meaningful abstraction, yet is unable to name.* This particular concept is an interesting case in point, since, I suggest, it has no name. I recognise, however, that words like 'inverse', 'reverse' and 'opposite' are commonly used in naming the sentence 'for all non-zero b, s(b)'. The Statutory Orders for the National Curriculum in England and Wales [Note 4.10] opt for "recognise that multiplication and division are inverse operations, and use this to check calculations". I'm not happy with this statement - perhaps this is pedantry on my part? - because it is commonplace to conceive multiplication and division as being binary operations (give me a pair of numbers, and I'll tell you their product). Thus, for me, the word 'inverse' implies, quite wrongly, that these ideas are set in the framework of a calculus of binary operations. But what we actually have (for every non-zero real number a) are two mappings (unary operations); multiplication *by a* and division *by a*; and now it *does* make sense to say that these are inverse mappings, under composition. It is well-known (Graham, 1992) that the authors of our National Curriculum were in too much of a hurry to think about such niceties.

This analysis demonstrates that *the beauty of the deictic 'it' lies in its function as conceptual variable. It (i.e. 'it') conveys the message, "I have something in mind. I know what I mean, and I think that you know what I mean". It can be a linguistic pointer to a shared idea, to an understood but un-named mathematical referent at the deep structure level.* It can give both of us secure and economical access to an algebraic proposition, whilst Susie sets about trying to put bounds on its generality.

The notion of a (possibly tacit) object of attention is sometimes captured by the term 'focus'; see e.g. Chafe (1972) for a linguistic account, Garrod and Sanford (1982) for a psychological one. Moxey and Sanford (1993) connect pronoun use to 'focus', in a way that confirms my view of 'it' as a linguistic pointer to concepts which occupy the attention of the speaker:

> focus can be inferred through ease of pronominal reference. Because personal pronouns such as 'it', 'she' and 'they' carry only minimal information to recover the referent [...] it is clear that in practice things in a discourse that that can be referred to by pronouns must be a small subset of the possible previous antecedents, otherwise ambiguity would be rife. For this reason focus has become closely associated with the conditions of felicitous pronominal anaphora [...]. The point is that pronouns are good for referring to things in focus, while noun-phrases of a more complex and informative kind are best for things not in focus. In this way ease or acceptability of pronominal reference can be used as *an index and a probe for the state of focus*, other things being equal. (p. 58, my emphasis)

Although she was not aware of it, Susie caused me to notice and to reflect upon the deictic use of pronouns in maths talk. As a consequence of our first four conversations, over a period of a month or so, I began to work on her deployment of 'it'. There were signs however, on my fourth visit to her school, that she was becoming irritated by my probing questions. I had asked her to explain her answer to the multiplication 8×32.4, which she had insisted on doing in her head. As she muttered to herself, I said:

S4:54 Tim: I hope you're going to explain this to me later.

I persisted until she said:

65 Susie: Help!... [pathetic tone] This will take *hours* to explain.

66 Tim: What will take hours to explain?

67 Susie: All of it.

Our conversation went into recess. When I returned, six months later, my attention shifted from 'it' to another pronoun.

ON 'YOU' AND GENERALISATION

I have already noted [S2:95] Susie's use of 'you' to mean 'I' in her statement "... and then you halved it". Whilst such a use of 'you' as a vague referent is familiar in adults as well as children, it would make perfect sense (but imperfect truth) if Susie had intended 'you' to mean her audience (i.e. myself). The ability to interpret the deictic element (pronoun in this case) is dependent on knowledge of the 'coordinates' (time, place, speaker, topic etc) of the deictic context. I discussed usage of the pronoun 'you' in the previous chapter. I shall now consider the particular significance of 'you' in mathematical conversation.

'You' can, of course, be used to address the person or persons to whom one is speaking. Such use is typically deictic but generally unambiguous. The following anaphoric examples come from longer transcripts in Brissenden (1988) of two groups of children using a computer program, *Trains*.

Craig: Nine add seven, that's sixteen, it's one ten and six. [but he enters 6 first and then 1]

Steven: Craig, you've got sixty-one now, it's the wrong way round ...

Gavin: Seventy-two plus seventy-two, that's a hundred and forty-four.

Teacher: Gavin, that was quick. How did you work it out?

In adult-child mathematical conversations where the power relationship is asymmetrical, *my transcripts indicate that the teacher/interviewer frequently uses 'you' to address the child, whereas the reverse is relatively rare.*

It is as though such usage of 'you' is a device which directly *points* to the other. And is it not rude to point? With the decline and disappearance of 'thou', English is almost unique among western European languages in having neither plural nor honorific distinction in second person pronouns. [Note 4.11] Brown and Gilman (1970) give an account of the power of pronouns as linguistic devices for expressing social distinctions in non-British European languages. French, for example, retains a pronominal social semantics in its 'T-V' (*tu, vous*) system: *tu* is the marked singular, expressing intimacy or certainly informality, occasionally condescension (to children by default). *Vous*, the unmarked plural, is also the singular for public or formal conversation, occasionally used as a marked pronoun of respect (or distancing with adults).

My data strongly suggests that the majority of instances of 'you' by children in mathematics talk can be seen to be indicative of things that happen "in the ordinary course of events" (Zandvoort, 1965, p. 128). Such things are *generalities*. Recall once more the pupil in Pimm's transcript:

Pupil: Is it that you square it - every time?

As Pimm observes, one pointer to the fact that the pupil is offering a generalisation is the expression "every time". *Another, I suggest, is the use of the vague, unmarked 'you', functioning as a vague 'generaliser'.*

In Chapter 3, I drew attention to this phenomenon in non-mathematical text. I shall proceed to give four examples of it from my transcript data.

Anna

In the extract which follows (dipping a toe into data which will be introduced in the next chapter) I am talking to two girls about ways of 'making' twenty. Anna (aged 10) has proposed 'minus one add twenty-one'. In my 'bookend' questions, probing for Roksana's position, she (Roksana) is the referent of my 'you'. But now study Anna's 'explanation speech', and consider "Who is 'you'?" for Anna.

T1:54	Tim:	Minus one add twenty-one. What do **you** think, Roksana? [pause] Right, explain to us why that would give us twenty, Anna.
55	Anna:	Cos nought add twenty equals twenty ...
56	Tim:	... right
57	Anna:	... so if **you**'re going into the minuses **you**'ve got to ... em ... **you**'ve, instead of saying twenty, that would equal nineteen, instead of twenty-one. And, minus, if **you**'re doing minus ... one add, add minus one, something equals twenty, **you** go minus one add twenty equals, it equals nineteen. So **you** need to go minus one add twenty-one equals twenty.
58	Tim:	Are **you** convinced by that, Roksana?

In choosing the impersonal 'you' in preference to 'I', the speaker has 'de-centred' and become, in some sense, detached from what s/he is asserting. Personal confidence - albeit tentative - in the general application of some process or proposition enables the speaker to offer it for others to appropriate. When Anna says [T1:57] "you go minus one add twenty equals, it equals nineteen" it could be that she is sharing some kind of number line imagery that (she believes) should be accessible and convincing to others. In any case she is saying that "anyone can do this".

Simon

I began the first of three mathematical conversations with Simon with an enquiry along the lines "give me two numbers whose sum is ten". I usually begin a first year undergraduate course in Number Theory with the same question: intended, in that situation, to explore what 'number' means to these new students. After similar preliminaries with Simon, he proceeded to determine the number of ways that any positive integer could be 'made' as a sum of two positive integers. It was many months later, following abortive trials with other tasks, that I decided to adopt 'Make Ten' for the next phase (Chapter 5) of contingent interviewing with a larger sample of children.

Simon progressed to consider making a given integer as a sum of three integers. First he worked on 20 and came to see that the number of ways is the 18th triangular number. (The details are in Appendix 3, transcript Si1.) Picking up the conversation at that point:

Si1:205	Tim:	Right. Suppose instead of twenty, right, I said how many different ways are there of adding up three numbers to make fifty.
206	Simon:	I'd do forty-nine times [pause] twenty-five.
207	Tim:	[pause] You'd better explain that.
208	Simon:	Um, no, I wouldn't. I'd do forty-nine times twenty-four.
209	Tim:	Explain it.
210	Simon:	Well, it's the triangular number of, em, it's the ... working out the triangular number of ... the forty-eighth triangular number.
211	Tim:	Mm-hm. Why do you know that?
212	Simon:	I'm going on the assumption that it works the same for twenty.
213	Tim:	What happens with twenty?
214	Simon:	It, em, I found the triangular number for eighteen, because ... the second number before twenty.
215	Tim:	Right, right. So what you do with fifty, you say ...
216	Simon:	Make, work out the triangular number forty-eight.
217	Tim:	Right.
218	Simon:	And to do that, I times it by ... so I do forty-eight times ... no, I do forty-nine times half of forty-eight, which is twenty-four.
219	Tim:	Right. Can you see why it's forty-nine times half of forty-eight?
220	Simon:	Yeh
221	Tim:	Why?
222	Simon:	Because, to work out a triangular number, you get the first and the last, and the second and that ...
223	Tim:	and multiply it by how much?

224 Simon: Um, the num... a half of the number ... of ... half the number of numbers you've got. So it's like from nought to forty-eight, so half of that, cos you've only got half the numbers to work out.

Observe in passing Simon's expression "it works" in Si1:212, strongly reminiscent of Susie's use of 'it' to point to a general relationship or procedure. In 206-212 he uses 'I' to describe what he did for 20 and what he predicts for 50. Whereas I use the vague generaliser 'you' in 215, Simon persists with the personal 'I' in 218 to describe how he would calculate a particular triangular number. In 222 and 224, however, he adopts the pronoun 'you' himself: here he is formulating a *general* procedure for such a calculation, and explaining why it works - Gauss' method, in fact, which I had in effect introduced him to earlier, for the purpose of finding the 18th triangular number (Appendix 3, Si1:189-203).

Susie

My fifth and final conversation with Susie was memorable, and is reproduced in full in Appendix 3, S5. This indeed was Susie's 'Kye' (Brown, 1981). The remarkable mathematical content deserves more extended attention, which it receives in Appendix 2. In this extract, Susie is developing a highly idiosyncratic method of dividing 100 by various fractions, five-sevenths in this instance.

S5:62 Tim: OK. Now, you said that it wouldn't work for seven-ninths didn't you, this method. Right? Now, I'd just like you to write down five-sevenths, just here.

63 Susie: I'm going to have to think though, very well. Um, I'll try ... [pause]. Ahh, of course ... [interrupted]

64 Tim: You have a think while I push the door up.

65 Susie: ... you can't ... I don't understand. It's definitely a hundred. So that means two ... Ahh, ahhh [*big moment*] you've got two left, and you need five each time. So if you have two hundred ... um ... divided by five. How many times does five go into two hundred? Well, it goes into one hundred twenty times ...

66 Tim: Mm-hm

67 Susie: Must go into forty times. So that's ... a hundred and forty.

Notice how Susie's "I" [63, 65] becomes "you" [65] after the "ahh" which seems to signify the moment of insight.

Again, the pronoun 'you' is an effective and non-trivial pointer to a quality of thinking.

Susie's shift from 'I' to 'you' signifies her reference to a mathematical generalisation.

The generality expression "each time" occurs in [65] as part of an account that has the quality of a generic example, for the exposition of the division method that Susie is developing

In my 25,000 transcribed words of mathematical conversations with 9-11 year-old children, 'you' is the most frequently-used word (744 occurrences), followed by 'and' (662), 'to' (400) and 'a' (394). In the "dividing by fractions" conversation with Susie above (2450 words), Susie and I each use 'you' forty times. *Every* time I use the word, I am addressing Susie, whereas she uses 'you' to address me *only twice* - once to make sure that she's not leaving me behind!

S5:187 Susie: And it has to be two hundred. So you would have two hundred [writes] divided by five. Do you understand that?

188 Tim: Yes thank you.

On the whole, *Susie reserves 'I' to mark her feelings and beliefs, or accounts of her personal actions, whereas 'you' indicates a kind of detachment* from her strategy and computational methods.

The same 'personal *versus* general' markers are evident in the final example.

Katy

In an undergraduate supervision with me [NT7], Katy is talking about her progress with a project on continued fractions:

NT7:17 Tim: And what have you proved?

18 Katy: Um, I've proved that ... I think, now I want you to have a look at it, 'cos I'm not sure if it's right, but I did this [...] yeah, I was trying to, I've had a look at, I said that, right, the root of A squared plus one is equal to A plus one over alpha, like we did before ...

19 Tim: Oh yeah, yeah, yeah, go on, yes.

20 Katy: Um, and it's also equal to A plus one over two A plus one over alpha because you can, because alpha goes on forever you can start it whenever you want.

21 Tim: Umm ... yeah, go on.

22 Katy: And so I put those two equal to each other.

The occurrences of 'I' are all set within accounts of personal actions located in time and space. This is also true of the 'we' in Katy's first turn [18]. I suggest, however, that the 'you's in her second turn [20] are located in some informal object language which is appropriate for the specification of procedures and algorithms - because 'you' is not any particular, actual person, rather it is anyone - which, of course, is why the pronoun functions as a vague 'generaliser'.

SUMMARY

My interest in pronouns in mathematics talk, and my observations about them in this chapter, derive from my belief in generalisation is the mathematical process *par excellence*. This conviction coincides with and relates to Mühlhäusler and Harré's view (1990, p. 58) of the primacy of the deictic role of pronouns. In this chapter, I have focused attention on pronouns in the mathematical discourse of novice users of the mathematical register. In particular I draw attention to:

- deictic use of 'it' to refer and point to mathematical concepts and generalisations which have not (or, for various reasons, cannot) be named in the discourse;
- the pronoun 'you' as an effective pointer to a quality of thinking involving generality; the shift from 'I' to 'you' commonly signifies reference to a mathematical generalisation.

The use of 'it' as a conceptual deictic enables the pupil to say what s/he could not say otherwise, to draw attention to mathematical entities whose name s/he does not know. The notion of 'focus' as locus of attention (Moxey and Sanford, 1993, p. 58) is important here, for the teacher who is sensitive to the pronoun/focus connection can be made aware of the presence of a cognitive focus involving generalisation.

The second person pronoun 'you' is a prevalent and effective pointer to generalities in mathematical discourse. This is perhaps especially true in children's discussion with their teachers, a context in which children rarely address the adult participant as 'you'. Like Delderfield's bank clerk (Chapter 3), pupils in mathematics classrooms need ways of distinguishing, for their audience, their experiences and feelings from detached observation and generalised objective comment. These two qualities coexist in mathematical activity, and both are necessary.

It would be interesting to know how and when young children begin to use 'you' in this impersonal sense; I am not aware of anywhere that this question is addressed in the literature. [see Note 4.5]

The study of generalisation continues in the next chapter, but the emphasis shifts to pupils' pragmatic means of conveying propositional attitude as they assert predictions and generalisations.

CHAPTER 5: HEDGES

Policeman: How many times did you hit him?

Ash: A couple of times maybe.

(*Casualty*, BBC1 TV, 5th November 1994.)

Mathematics, viewed as a field of human endeavour, as opposed to an inert body of knowledge, offers considerable potential for intellectual risk-taking. To participate in this endeavour we are obliged to lower defences, and in speaking, to expose ourselves. Nothing ventured, nothing gained; encouraging pupils to take risks, in the form of making predictions and conjectures, might be expected to be a feature of effective teaching. At 'low levels' of mathematical activity - recall of facts, rehearsal of algorithms - the risk is low, at least for pupils for whom a history of success has cultivated confidence. Not so, of course, for those who have suffered ridicule at the hands of insensitive teachers and inconsiderate peers. Mathematics holds an invidious reputation within the school curriculum, being associated with fear of error and consequent public humiliation.

One common perception is that the questions teachers ask their pupils are not searchlights focused to reveal truth, but traps set to expose ignorance. Janet Ainley, studying children's perceptions of the purposes of teachers' questions, calls such uses 'testing questions':

> Because testing questions are so common, particularly in mathematics where answers are seen as being clearly 'right' or 'wrong', there is a danger that pupils may perceive all teacher questions in this way. Such a perception would inevitably be detrimental to attempts to encourage discussion, investigative work or problem solving in mathematics: pupils will feel that the teacher always knows the 'right' answers to any questions she asks, and furthermore that the teacher is always judging pupils by the answers they give. It is not surprising that pupils are reluctant to risk giving 'wrong' answers in these circumstances (1988, pp. 93-94).

From a broader perspective of social discourse, Labov (1970) comments that a question is normally deemed appropriate only when the enquirer meets certain speech act 'sincerity' conditions - including, in this case, that s/he doesn't know the answer, would like to know it, and has reason to believe the hearer is able to supply it. Labov shows that questions in classroom situations are exempted from these rules, and that the conditions governing appropriateness in the answers to such questions differ accordingly. The default expectation of pupils is that 'teachers' questions' are what Labov and Fanshel (1977) call 'Requests for Display'. These differ from other questions in that the enquirer (A) already has the information sought in the question, and "the request is for B to display whether or not s/he has the information" (p. 79).

Laurie Buxton has addressed the issue of success and failure in the mathematics classroom from the emotional perspective of the learner:

> Most classroom maths sets tasks, often with very clearly defined goals; whether they have been reached or not is seldom in doubt [...] This clarity tends to enhance the sharpness of emotional response. There is a nakedness about the success or failure in reaching a goal that evokes clearly defined emotions whose nature one cannot disguise to oneself. (1981, p. 59)

Anne Watson recognises how, in time, this can generate inhibitions in pupils, who:

> are worried about being wrong and nervous about asking for help if "being wrong" and "needing help" have, in the past, been causes of low self-esteem by leading to ridicule, labelling or punishment. (1993, p. 6)

This tendency is illustrated in the words of two 16-year-old girls (C and S) who had chosen not to continue mathematics studies at school. Here, they are interviewed by Susan Hogan, who asks about their experience of 'speaking out' in mathematics classrooms:

SH: And do you need to be confident in order to speak ... ?

C: Yeah, because there were people in the class that were so good that you kind of ...

S: ... thought well, they're gonna laugh at me if I get it wrong.

(From a research student presentation at the Open University, 11th November 1995).

Whilst Buxton speaks of nakedness, the word that comes to me is vulnerability. Not, of course, a state of being in danger of physical offence, but an exposure to intellectual injury. How odd, how unfair it is, that the 'crime' is cognitive, but the penalty affective. Extrinsic and intrinsic sanctions are associated with being wrong; it follows that there is a high premium attached to being right, with insufficient acknowledgement by pupils and teachers that uncertainty is a common and valid, indeed an honest and honourable, state to be in. One could go further, and insist that uncertainty is a productive state, and a necessary precondition for learning. For once we believe we 'know', we are no longer open to the possibility of further knowing. When mathematics is coming into being in the awareness of an individual, uncertainty is to be anticipated and expected. This is the essence of Mason's 'conjecturing atmosphere'.

> let it be the group task to encourage those who are *unsure* to be the ones to speak first [...] every utterance is treated as a *modifiable conjecture!*' (1988, p. 9; emphasis in original)

The absence of such an atmosphere in the experience of the students interviewed by Hogan (above) is evident:

SH:	Would you ever volunteer an answer in class [...]?
S:	Yes, if it's something I definitely know the answer to I'll put my hand up and say the answer.
SH:	And if you're not sure?
S:	I don't - I wait for someone else.

Yet, in the making and learning of mathematics, uncertainty ought to be expected, acknowledged and explicit. All learning - in the sense of coming to know something that was formerly unknown - involves some kind of act of commitment on the part of the learner. There are, for example, ample testimonies to excruciating reluctance to concede the invariance of the cardinality of small, finite sets (Fielker, 1993), as well as resistance to belief in Newton's second law of motion (Orton, 1985). It is also a different matter to assert your version of truth as opposed to assent to someone else's. To assent may, but need not, entail full commitment, whereas to assert is to commit oneself, and so to risk accusation of error.

In particular, generalisation, as a manifestation of inductive reasoning, has been described (Chapter 1) as a kind of cognitive leap, embracing enthymematic premises that cannot be *known* to be true at the moment when a conjecture is made. To assert, to articulate such a conjecture is indeed a risky business. An investigational approach to learning - nurtured by the ATM in the 1960s, approved by Cockcroft in 1982 before being adopted and tamed by the GCSE examining boards - is by no means universally welcomed by pupils, many of whom feel that the risk involved is too high a price for the *frisson* of discovery. The articulation, the assertion of the predicting or generalising insight - making public what it is that one has 'seen' - can be shot through with uncertainty until it meets with approval:

> Investigations have to be 'managed' sometimes, as in 'I managed to solve the problem'. [...] It is worth asking about students who have *not* managed their

> anxiety. Managing an investigation certainly involves not only managing the technical and mathematical tools, but the affective components too. Solving problems and investigating situations (and even mastering conventional mathematics) are risk-taking activities and require courage as well as skill. It would be good if some writers (without being sentimental about it) would give some attention to these often unspoken aspects. (Wheeler, 1984, p. 25)

In a quasi-empirical environment for mathematics teaching and learning, the process of 'coming to know' is an exhilarating ride - at times risky, infused with uncertainty, at other times replete with "approaching completeness" (Watson, 1995). In this chapter, I shall describe and analyse some ways in which uncertainty is coded in spoken language, with reference to a mathematical study carried out with children aged between 10 and 12. I chose to work with them on a task which required (amongst other things) prediction, generalisation and explanation. A linguistic class - **'hedges'** - of pointers to such moments of uncertainty will be identified. My aim is to draw attention to the presence of such hedges in pupils' mathematical discourse, and to analyse how speakers *do* things with them.

METHOD

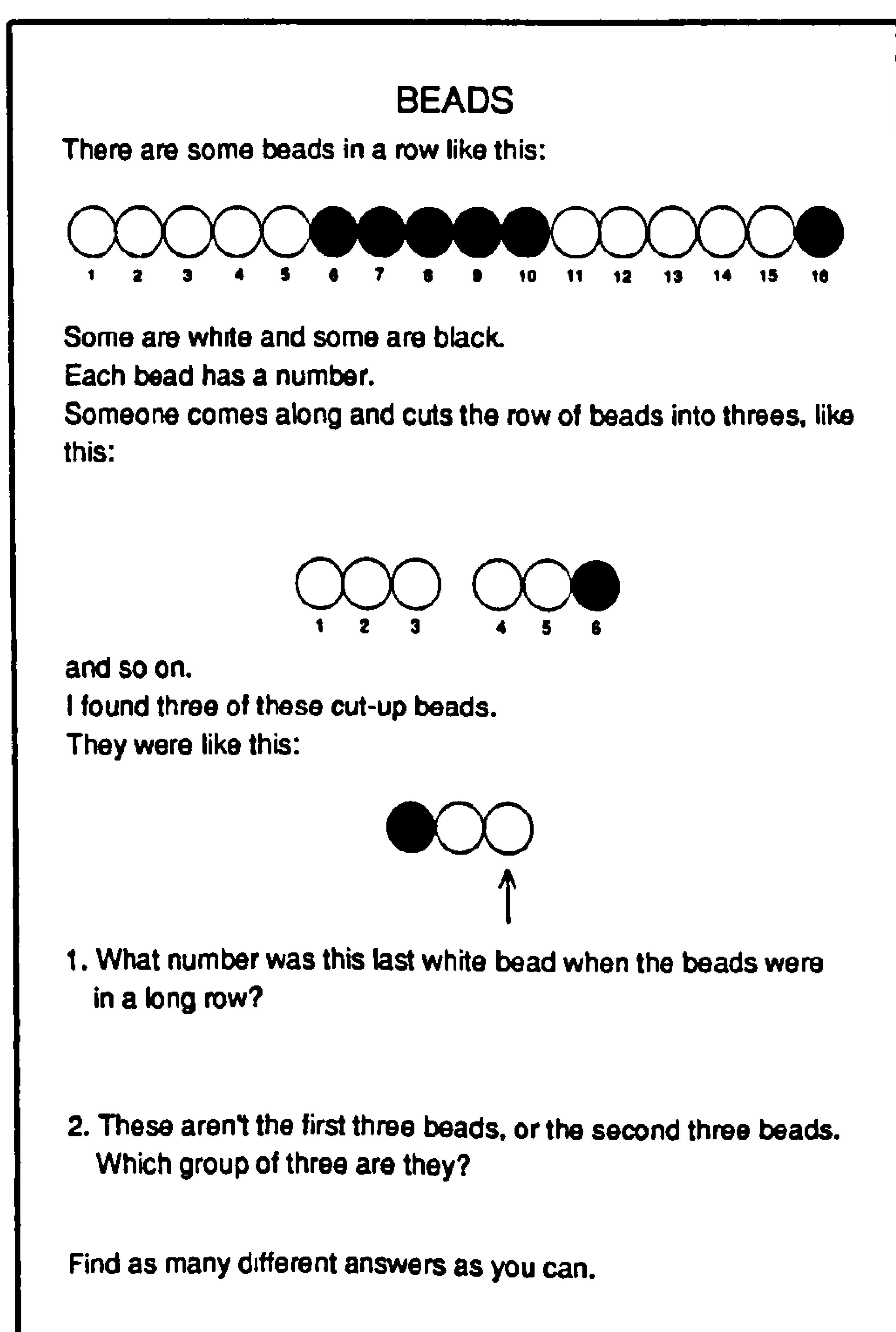
The study described here arose from my interest in the language that children (specifically those in the 9 to 12 age range) use to invoke, describe and engage with the mathematical process of generalisation. In Chapter 4, I reported how one articulate 9-year-old girl, Susie, deployed the pronoun 'it' to refer to concepts and generalisations for which she had no name, or for which no received name existed. Another pronoun, 'you', was shown to be associated with the enunciation of general procedures and relations. Subsequent extended mathematical conversations with Simon, aged 12, served to confirm these observations and develop my expertise as a 'contingent' interviewer. In this study, I moved from extended case study to replication of an 'experiment' with a number of children. Intending to diminish the part I played in the discussion, I worked with ten pairs of children aged 10 or 11, for about 30 minutes with each pair, and encouraged some peer interaction on a common mathematical task. I did, however, remain as a participant, as opposed to a passive observer, originally so as to maximise their engagement with the mathematical task. [Note 5.1] Contrary to my original intentions, I ended up reflecting on the pupil-interviewer interaction too. This experience parallels that of Paul Cobb and his collaborators (1992) as reported in their account of a teaching experiment with a class of second-grade children.

we initially took a radical constructivist position [...] and focused almost exclusively on students' learning from a cognitive perspective. Only later did we widen our purview to encompass an interest in the teacher's learning and classroom social interactions, and begin to complement our initial constructivist position with constructs derived from symbolic interactionism [...] and ethnomethodology. (p. 101)

The interview technique which I deployed is a mildly restricted form of 'contingent questioning' (Ginsburg, 1981; 1983) which I described in Chapter 0. Recall two of the features of the contingent interview, a method which:

- employs a task to channel the subject's activity;
- has some degree of standardisation.

Choice of task: I compiled a number of tasks and considered their competing merits in relation to certain requirements - the chosen task would be replicated with ten pairs of children, I could not ditch it mid-stream. It had to be accessible mathematically to most 10-year-old children; have some intrinsic interest for the them; have definite potential


BEADS

There are some beads in a row like this:

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

Some are white and some are black.
Each bead has a number.
Someone comes along and cuts the row of beads into threes, like this:

1 2 3 4 5 6

and so on.
I found three of these cut-up beads.
They were like this:

1. What number was this last white bead when the beads were in a long row?

2. These aren't the first three beads, or the second three beads. Which group of three are they?

Find as many different answers as you can.

Figure 5.1

for stimulating prediction and generalisation; preferably be accessible in terms of their explaining their generalisations; and reasonably expect to lead to some worthwhile (if unpredictable) mathematical outcomes in about half an hour.

The first such task that I trialled I call 'Beads'. The essence of the task is shown in Figure 5.1.

Trials with three pairs of children aged 10-11 indicated that it was an accessible task for them, and that one solution could be found fairly quickly. Unfortunately, however, they found it difficult to arrive at a second solution without a good deal of prompting from me. Consequently, my share of the 'conversations' was greater than I would have wished. Yet this was hardly a task where "a single instance" (Mill, 1873, see Chapter 1) would be likely to be "sufficient for a complete induction"; no prediction or generalisation was possible without a second or third solution. It also proved impossible to arrive at a satisfactory termination of the conversation in half an hour. Whilst the task is apparently about something 'real', a mathematical analysis might describe it as an application of the Chinese Remainder Theorem for two linear congruences. The cognitive demands are indicated by the presence in the situation of 'multiple interacting systems' - the need to hold in place and satisfy two competing requirements of a solution - which Lunzer (1978) identifies as requiring formal operational reasoning. It was the tendency of the children to neglect one of these competing requirements that necessitated my constant intervention for prompting in the discussion. After one session I explained what I was trying to do ("I'm interested in the things people say when they're talking about maths") and asked what they thought about 'Beads'. Insiya evaluated the task as "Quite interesting, but quite difficult ... not too difficult". Sarah said "You'd have to do it with people who knew their three times table".

In the end, I abandoned 'Beads' in favour of an easily accessed combinatorial problem which I call 'Make Ten'. I have already described in Chapter 4 how this idea originated in my teaching and was piloted in one of my earlier weakly-framed conversations with Simon.

Standardisation: The mathematical encounter with each pair of children was standardised to the extent that I intended each 'Make Ten' interview to proceed proceeded in five 'phases' which I had planned in advance.

I initiated each interview with the same conversational gambit [Note 5.2] - a combinatorial problem which will be familiar to primary school teachers, although its potential as a starter for making generalisations may not. It begins with consideration of the number of ways that 10 can be 'made' as a sum of two 'numbers'. I did not work from a 'script', but the following opening (with two boys, Jubair and Shofiqur) is typical. [Note 4.2]

T5:1	Tim:	Jubair, I'd like you to give me any two numbers that add up to ten.
2	Jubair:	Six add four.
3	Tim:	Six add four. Shofiqur?
4	Shofiqur:	Eight add two.
5	Jubair:	Five add five.
6	Tim:	OK, so you get the idea. Now what I want you to decide between you is . how many ways is it possible to do that?

The next three exchanges were not at all typical, however:

7	Shofiqur:	[almost instantly] Nine ways.
8	Tim:	Nine ways.
9	Jubair:	No, ten ways.

It was more usual for the children to list the possible sums, orally or on paper, and then to count them. Then I would say something like:

> Now just as you eventually decided about that question for ten, I'd like you to decide between you how many different ways are there of doing that for twenty?

The first phase of the interview was planned to proceed as above with similar examples of listing sums and counting how many had been found. I would propose the numbers to be 'made' in this way, my choices depending on the children's earlier responses to my questions about 'making' 10 and 20 - in particular, on the facility they displayed and whether reversals such as 2+8 and 8+2 were both counted. [Note 5.3 summarises the mathematical consequences of such choices.] The next phase would then involve my proposing a further target number - say 30, 50 or even 100 - slightly out of the range of those already counted, and inviting a prediction of the number of ways this number could be made.

T3:60	Tim:	OK. Now a funny question. Supposing the number now that I'm interested in is twenty-three? So you've got twenty-one add two, and so on. How many ways?
61	Caroline:	Twelve, would it?

62 Alex: Yeah, probably.

63 Caroline: Twelve.

64 Tim: What makes you think twelve Caroline?

Subsequent phases, contingent on preceding ones, would involve my probing for the thinking behind this prediction (and possibly others) and discussion of perceived 'rules' - conjectures about what might happen with 'any' number. For example:

T4:138 Tim: Right, OK. Is there a kind of rule that you could state generally, I mean supposing I now picked out any number ... you know like five hundred and thirty-seven or something ... and said how many ways can you make that from adding two numbers. How would you know what the answer was?

139 Alan: Just take away one, and then you'll know how many you can get. It's the same here, ten, there was nine possibilities, twenty, there was nineteen possibilities, thirty-seven there was thirty-six possibilities.

In some cases, we continued to test the generality of such conjectures, and tried to see why they might be true 'in general'. In practice, such proofs were always founded on the possibility of 'seeing the general in the particular' (Mason and Pimm, 1984), producing confident awareness of how things would be for any other particular, as it were. That is to say, the pupils explained their generalisations by accounts of generic examples (Chapter 1). One instance of this (a lengthier quotation from Alan and his partner Harry) is given at the end of this chapter.

Restriction: In an extreme, weakly-framed form of contingent interviewing, no contribution is deemed irrelevant; the interviewer follows the thoughts of the interviewee and encourages him or her to develop them further. In this sense the subject determines the agenda and is free to deviate from it or to redefine it. My extended conversations with Susie and Simon were of this kind. The 'Make Ten' interviews were intended to generate some data as a basis for prediction and generalisation. The interviews were contingent in the sense that the children were given freedom to decide what kind of sums would be eligible, and which sums would be counted as 'different'. On the other hand, I expected to shape each conversation as indicated above, aiming for prediction and generalisation.

The conversations were audio-taped and transcribed, providing a corpus of some 24,000 words (both the children's and mine).

'MAKE TEN': FRANCES AND ISHKA

The following extracts give a fuller picture of how conversations arising from the task developed. They are from an interview with two girls, Frances and Ishka, both about 10½ years old. The interview transcript is in Appendix 3, T6. Here, I have separated that transcript [T6] into five 'episodes' A to E. These 'episodes' are thus the realised equivalents of the planned 'phases' of the interview. For most of the interviews the phase-episode correspondence was fairly close. The episodes of T6 are summarised and illustrated by the following extracts.

EPISODE A [7½ minutes] In this phase I introduce the problem, that of making 10. The children list and decide 5 ways, allowing no reversals (decided by Frances).

T6:16	Frances:	Four and six, five and five, six and ... oh that's the same.
17	Ishka:	Five ways?
18	Frances:	Maybe.
19	Ishka:	Mm, maybe ... I think ...
20	Frances:	What do you think?
21	Ishka:	We haven't had five five have we?
22	Frances:	We have!
23	Ishka:	Oh OK, erm ...
24	Frances:	The others are like if you do six four, we've already done four six.
25	Ishka:	Mm [sighs]
26	Frances:	Shall we just say five ways?
27	Ishka:	There's about five.

I then ask about making 20. They list and decide 10 ways. Finally, I ask about making 13; Ishka lists and decides 6 ways.

EPISODE B [2 minutes] I begin by asking them to recall results so far. Then I ask about making 30, inviting an initial prediction. Frances predicts 15, Ishka agrees, and explains how her prediction relates to the earlier results. There is an air of plausibility rather than certainty in their attitude to Frances' prediction.

T6:105	Ishka:	I think there'll be around ...
106	Frances:	Fifteen?

107 Ishka: Yup.

108 Frances: Maybe? [...]

113 Ishka: Most of them are half or just about one away from ... [...]

117 Tim: OK. Prediction of fifteen. Er, so ... lets just go back to what Ishka was saying. She was saying that in most cases it's about half.

118 Ishka: Well, yes, 'cause ten was five.

119 Tim: Right. [...]

120 Frances: Twenty was ten. [...]

123 Frances: Thirteen was about six.

124 Ishka: But, erm thirteen was six.

125 Tim: OK.

126 Ishka: Although that isn't exactly half.

I proceed to ask about making 100. Frances instantly answers "Fifty?", Ishka agrees in a vague way, but conveys considerable uncertainty when pressed to commit herself to Frances' prediction.

129 Frances: Fifty?

130 Ishka: About fifty yeah.

131 Tim: About fifty. Now are you saying about fifty, Ishka, because you're sort of playing safe or I mean do you really think it is fifty?

132 Ishka: Well maybe not exactly, but it's around fifty basically?

133 Tim: OK. And Frances do you think it's exactly fifty or around fifty?

134 Frances: Maybe around fifty.

EPISODE C [3½ minutes] I return to making 30 - how sure are they that there are 15 ways?

135 Tim: OK. What about this prediction of fifteen for er thirty, was that around fifteen or exactly fifteen?

136 Ishka: It's fifteen or around.

137 Frances: Yes.

138 Ishka: 'cause we can't be exactly sure until we've tried it, but ...

The girls list and count 15 ways.

EPISODE D [1½ minutes] Looking for some structure in the list for 30:

194 Tim: [...] you started at the top and went one, two, three, four, five ... can you notice something about these numbers that would immediately tell you there's fifteen without writing them all down?

195 Frances: There's one to fifteen, so fifteen numbers?

Then imagining what a list for 100 would look like:

198 Tim: Right. If you did a similar listing for, um [...] a hundred [...] If you did a list like this and you started at ... what would be the two equal numbers at the top?

199 Ishka: Fifty and fifty.

200 Tim: Fifty add fifty, and then it would go forty-nine [...]

209 Tim: And how many numbers would there be in those, or how many ways would you have done it?

210 F+I: Fifty.

EPISODE E [4 minutes]

Articulating general 'rules' for the number of ways:

212 Ishka: So the even numbers are exactly half.

213 Tim: That's really good Ishka.

214 Ishka: And the odd numbers are ...

215 Frances: Are exactly half.

216 Ishka: Are sort of, are one off ... that.

Generic examples suggest proofs of the rules:

244 Tim: Right, OK, I just wonder whether it's possible by writing down all the ways of getting, say, thirteen like this ... yuh? ... whether you could then see why it was half of the number before. [...]

245 Frances: Yes.

246 Tim: Do you want to try it Frances?

247 Ishka: So we start at half of thirteen.

248 Frances: Half of thirteen would be ...

249 Ishka: Six.

250 Frances: Six and seven. Six point five ... [laughs]

251 Tim: You can't have six and a half and six and a half, so what will you start with?

252 Frances: Six and seven.

We discuss why there must be six sums in the list: the interview concludes.

HEDGES AND UNCERTAINTY

In time, and through discussion with others, I have become convinced of the significance of one particular surface-level feature of the data. This is the children's use of the category of words which Lakoff named 'hedges' (Chapter 2) - examples of which include 'about', 'around', 'maybe', 'think', 'sort of', 'normally', 'suppose', '(not) sure', '(not) exactly'. The extracts from the interview with Frances and Ishka give an authentic and reasonably sequential picture of when and how they are used. Such words convey a sense of vagueness, at times of uncertainty - a state of mind which, as I have already observed, one would expect to prevail in a conjecturing moment. I shall also have something to say about the way that I too (as teacher/interviewer) use hedges in the discourse.

To speak (as I have) of *using* language to *convey* states of mind suggests intention on the part of the speaker, that the effect on the hearer is not just accidental. Human beings have a set of wants and needs to do with self-esteem that they seek to satisfy (for themselves or others) in interaction with others. Language use is one means toward that end. By this, I don't mean that speakers consciously choose words (including hedges) to achieve an effect - although some do - but rather that they have learned to use them for that purpose. In any case, I suggest here that conveying [implicating] uncertainty is the intended purpose, and not just the effect, of their hedging.

Whilst the focus of this chapter is the nature and function of hedges in prediction and conjecturing activity, the use of 'Rounders' (a subset of hedges usually associated with lack of precision) such as 'about', 'approximately' and so on is to be expected as a linguistic feature of *estimation*, as a device to indicate that the speaker is providing as much accuracy as is possible or appropriate in a given situation (Channell, 1994).

Estimation, prediction and generalisation are all mathematical processes which, to a degree, involve some element of uncertainty. In fact, Clayton (1992) has studied estimation as a "risk taking" activity. In talk about mathematics, children may convey uncertainty with various degrees of subtlety, and with various pragmatic purposes, through the use of hedges. Whether or not fear, anxiety and so on are present in those situations must depend on the spirit in which the mathematics learning takes place. This is to some extent determined and controlled by the teacher - by the way that s/he responds (language of word and body) to pupil's contributions. The willingness of schoolchildren to expose their thinking will depend on whether or not teacher and pupil share a belief or explicit agreement that they are working in a 'conjecturing atmosphere'.

HEDGE TYPES IN MATHEMATICS TALK

Each of the four categories of hedge introduced in Chapter 2 is in evidence in the Make Ten transcripts, and is associated with particular kinds of goals. Here I take a first look at the pragmatics of these categories in mathematics talk, with some examples from Make Ten.

Plausibility Shields are typified by 'I think', 'maybe' and 'probably', as in this excerpt from the episodic overview of the Frances/Ishka 'Make Ten' interview.

T6:16 Frances: Four and six, five and five, six and ... oh that's the same.

17 Ishka: Five ways?

18 Frances: Maybe.

19 Ishka: Mm, maybe ... I think ...

A Plausibility Shield implicates (in the Gricean sense) a position held, a belief to be considered - as well as indicating some doubt that it will be fulfilled by events, or stand up to evidential scrutiny. It is a means of offering an idea without the obligation of commitment to its truth.

The second kind, an **Attribution Shield,** implicates some degree, or quality, of knowledge to a third party. In the 'Make Ten' data there are relatively few Attribution Shields, and these tend to be used by me rather than the children, as a teacher-like device for meta-comment (Pimm, 1992) on the activity. Thus, with Kerry and Runa:

T8:176 Tim: OK. Um, how many ways would there be then for twenty-four?

177 Runa: Twenty-four? Add?

178 Tim: Same kind of thing, but with twenty-four.

[Kerry is whispering, seems to be counting something]

179 Runa: Um, nineteen. Nineteen ways.

180 Tim: Nineteen ways, says Runa.

181 Runa: I just guessed.

182 Tim: Kerry's still thinking.

183 Kerry: Ten.

Here, I use Attribution ("says Runa") in T8:180 as a ploy for being non-committal about the contribution of one child, in order to obscure my evaluation of her answer [T8:179], and to encourage the participation of the other child.

The semantic effect of Approximators - the second major category of hedges - is to modify (as opposed to comment on) the actual proposition, making it more vague. It was observed in Chapter 2 that speakers make propositions vague in this way to fulfil all kinds of purposes (Channell, 1994, pp. 173ff), one of which is "giving the right amount of information" i.e. as much as is needed in the context of utterance. For this reason, I might tell you that the time is half past three (incidentally concealing the Approximator 'about') when my watch says 15:28:22. Another possible motive for being vague discerned by Channell (1990, p 98) is "downgrading the importance of something so as to highlight something else". For example, the proposition

> The number of seconds in a year is about pi times ten to the power seven.

deliberately sacrifices precision (which could be improved) in order to draw attention to a pleasing numerical coincidence. 'About', 'around', and 'approximately' are examples of **Rounders**, which constitute the first subcategory of Approximator. Rounders are usually associated with estimation in the domain of measurements, of quantitative data. Association with prediction and generalisation does not readily come to mind, yet Rounders occur frequently in the Make Ten corpus, to qualify combinatorial prediction, as in Episode A with Frances and Ishka:

T6:26 Frances: Shall we just say five ways?

27 Ishka: There's **about** five.

and again, in Episode B

T6:105 Ishka: I think there'll be **around** ...

106 Frances: Fifteen?

107 Ishka: Yup.

Adaptors, such as 'a little bit', 'somewhat', 'fairly', attach vagueness to nouns, verbs or adjectives. The following examples are from the Make Ten interview with Jubair and Shofiqur. Shofiqur has just indicated what a list of ways of 'making' 20 would look like, and predicts 21 different ways.

T5:66	Shofiqur:	... It's just **a bit** the same, like this [indicating the list for 10].
67	Tim:	So Shofiqur is **pretty** convinced that it's twenty-one. Right, are you persuaded by his argument?
68	Jubair:	Not **really**.
69	Tim:	Have a go at - I'm **fairly** convinced what you said Shofiqur, have a go at convincing Jubair that there are twenty-one ways. I mean, take it slowly.
70	Jubair:	Come on then!
71	Shofiqur:	I only took a guess.

Adaptors suggest, but do not define, the extension of categories, concepts and so on (see how I just did it with 'and so on'). Thus, Shofiqur uses an Adaptor phrase 'just a bit' with respect to same(ness); I use two Adaptors here, 'pretty' and 'fairly', to suggest, first that Shofiqur's conviction, then mine, is not simple and unreserved, but of a fuzzy kind.

A sift of the transcripts suggests that it is I, rather than the children, who make most use of Adaptors. Like Attribution Shields, and for similar reasons, I use them as a means of commenting on the children's contributions. Specifically, I use them to make indirect comments on their predictions, generalisations and explanations.

The taxonomy provides a setting for studying the significance of the various hedges used in my Make Ten interviews. The framework is useful in making distinctions and providing starting points. Whilst the four categories of hedges are sufficient (in the sense that they embrace the hedges in my data), they are not disjoint. Bear in mind that in conjectural mathematics talk there is an affective subtext just below the surface of the propositional text. It is there because mathematics is a human activity: the participants care about the mathematics, but they also care about themselves, their feelings and those of their partners in conversation. In the next section I shall show how, on occasion, speakers use Approximators for Shield-like purposes. There is a good case, in fact, for speaking of shield*ing* and approximat*ing*, to emphasise the effect of hedges in the context of use, as opposed to the identification of some rigid lexical categories.

PARTICULAR HEDGES IN MATHEMATICS TALK

The children whom I interviewed were aged 10 or 11, and were being invited to make mathematical predictions and generalisations. With reference to the 'Make Ten' transcript data, my central claim will be that when they hedge, it is more often than not in order to implicate (in the Gricean sense) uncertainty of one kind of another. In other words, their hedges predominantly are, or have the same effect as, Plausibility Shields. Later, I shall suggest that Shields are deployed at significant and identifiable stages in the interviews. Furthermore, I will show that the teacher/interviewer (me) also hedges, but typically for different purposes. These teacher-like purposes - to which intentions, in this case, I have relatively direct access - will be considered from time to time.

For the sake of maintaining coherence in the argument, whilst sampling from the data, I shall examine when and how particular hedges, or small groups of hedges, are used.

'maybe', 'think'

I have already observed that' maybe' and 'think' are stereotypic Plausibility Shields which can successfully convey a speaker's lack of full commitment to a proposition under consideration. It is necessary here to give more detail from Episode A of the Frances/Ishka interview, for immediate and future reference. I had asked the two girls to come to an agreement about the number of ways of making 10. Their discussion proceeds:

T6:12	Frances:	There's one and nine.
13	Ishka:	Yeah.
14	Frances:	So that's one. Two and eight ... and then there's
15	Fra & Ish:	Three and seven.
16	Frances:	Four and six, five and five, six and ... oh that's the same.
17	Ishka:	Five ways?
18	Frances:	Maybe.
19	Ishka:	Mm, maybe ... I think ...
20	Frances:	What do you think?
21	Ishka:	We haven't had five five have we?
22	Frances:	We have!
23	Ishka:	Oh OK, erm ...

24 Frances: The others are like if you do six four, we've already done four six.

25 Ishka: Mm [sighs]

26 Frances: Shall we just say five ways?

27 Ishka: There's about five.

28 Tim: Erm, I'd like you to be more convinced Ishka. I mean if it's about five then it's four or six or seven or whatever ... the number's sufficiently small that I think you should be sure one way or another.

29 Frances: I think it's five ways.

30 Ishka: But I'm sure.

31 Tim: You are sure.

32 Frances: Me too.

Having enumerated five ways, Frances begins to repeat herself [T6:16] - "oh ... that's the same". Rather, she offers me (and Ishka) the first insight into what sameness means to her in this context. She has an implicit criterion, which surfaces when she withdraws "six and ...". Ishka evidently shares or accepts the view that reversals will not count separately, and she asserts [T6:17] that there are five ways.

The fact that Ishka's claim is tentative is indicated by rising intonation ("Five ways?"), which transforms her statement (that there are five ways) into a question. This is one of a number of instances (in Make Ten and elsewhere in my data) where it could be claimed that statements are hedged with rising intonation. Such prosodic hedges (the linguistic term 'prosody' refers to variations in pitch, loudness, tempo and rhythm) are effectively Shields. This issue is discussed again at the end of the next chapter.

Frances [T6:18] perhaps echoes Ishka's uncertainty; or perhaps she may feel that Ishka's answer is offered prematurely, before she has exhausted all the pairs she can bring to mind. In any case the pair now seem to have an understanding that it will be productive to assert their uncertainty, and reconsider the "five ways" claim. Ishka effectively conveys this [T6:19] in the form of two Shields without a substantive proposition. Frances encourages her ("What do you think?") to articulate her position - this is typical of a number of instances of apparent teacher-like behaviour by Frances in Episode A, at which phase of the interview she projects herself as the dominant, more confident partner. However, having encouraged Ishka, she is impatient at Ishka's next contribution, which only suggests that Ishka has forgotten what has already been listed. Frances indicates [T6:24] that she is now satisfied that no further possibilities have been overlooked. [Note 5.4] There follows [T6:26, 27] an apparent reversal of the earlier roles [T6:17, 18] of Ishka and Frances in relation to the claim that there are five ways. One each occasion one has sought agreement with a hesitant assertion that there are five ways; the other has given hedged assent - Frances with a Shield, Ishka with a Rounder. The second time [T6:26, 27], however, I inferred that Frances was fully committed to the claim, whereas Ishka was not - "I'd like you to be more convinced, Ishka". I had, after all, introduced the dialogue cited above with a clear request for common consent:

T6:10 Tim: I'd like you two to agree between you ... incidentally we'll adjust that [microphone] Frances so it's not quite so close, right, um I'd like you just to - yours is fine Ishka - I'd like you two to agree between you, how many different ways there are of doing that. Right? Two numbers that add to ten, and I'll just be quiet for a moment.

My repeated request for agreement is complied with by Frances and Ishka to a remarkable degree, certainly in comparison with most of the pairs I interviewed for Make Ten. Ishka is not yet, however, prepared to concede unqualified agreement [T6:27]. The function of her chosen hedge, 'about', will be considered later in this chapter. In any case, she has successfully implicated the fragility of her commitment, borne out by the fact that I press her quite explicitly on the matter of being (more) convinced, urging that she "should be sure". Frances responds with an apparently hedged (but see my discussion of the ambiguity of 'I think' later) indication of where she stands. Ishka's response is unhedged, fully committed - but is it genuine, or have I blackmailed her into renouncing doubt in order to please me? After all, what I have demanded is not 'the answer' but for Ishka to be "more convinced". Of course, I really wanted both! I am at this stage of the interview encouraging the children to generate valid instances of a generalisation-in-waiting. That there are five ways of making ten is such an instance. I readily accept Ishka's assurance that she is "sure" without comment as to whether or not she is right.

'Maybe' is a modal form which seems to be user-friendly, in that it is favoured by the children in comparison with the apparently synonymous 'perhaps' and 'possibly', which occur not once, in the children's speech or mine, in the whole corpus. The following transcript data illustrates the appearance of 'maybe' within hedged predictive statements:

T9:133 Tim: Alright um, supposing, we've done, we've done ten, twenty, thirty, sixteen. [...] I'd just like you to sort of say how many you think there would be, say for the number twenty four. [...]

136 Rebecca: [...] twenty-two? No, not twenty-two ways. Twelve ways? [...]

140 Rebecca: Oh yeah, on the twenty there were more ways than the sixteen, so on twenty-four ... must be more then the twenty, because that was less, because it was a lower number.

141 Tim: Right, how many more?

142 Rebecca: I'm not sure.[pause]

143 Runi: [whispers] Eleven and twelve, [inaudible, presumably "forty"] ways.

144 Rebecca: Not forty, fourteen

145 Runi: Yeah, that's what I was going to say

146 Tim: Let's just see. Runi thinks **maybe** fourteen ways, and I think you suggested twelve Rebecca, yeah?

147 Rebecca: Yeah.

148 Tim: Um, what was your reason for suggesting twelve?

149 Rebecca: Well, it was four off than twenty and then twenty-two [?twenty ... two] was two less than four so you've twelve. Have twelve because, if you had, twenty had ten ways and twenty four was four more than twenty then **maybe** it would be twelve because it's um ... half way in between.

150 Tim: 'cos it's half way in between. OK. And what do you think Runi, are you saying it's four more, so it's four more ways?

151 Runi: Yeah, that's what I was thinking of.

When Rebecca explains her reason for suggesting twelve (as I put it), she seems to be reasoning that what happened in the increase from sixteen to twenty might happen again with a further increase to twenty-four. But, I argue, she is signalling an awareness that she might be jumping to conclusions by hedging [T9:149] "maybe it would be twelve". It is an honest and straightforward expression of doubt, as to the validity of the reasoning and the conclusion. By contrast, I double-hedge in [T9:146] "Runi thinks maybe .." as a device to cast doubt on Runi's unhedged - and incorrect - contribution ("fourteen ways") which is beginning to take over from Rebecca's interrupted - but correct - train of thought. We are some way into the interview, this is the fifth example I've asked them to consider, and I'm getting impatient. My reaction to Runi's off-course prediction is to undermine it by attributing doubt where there may have been none. Thus, my "Runi thinks ... " [T9:146] is intended to implicate "but that's only what Runi thinks". Furthermore, "Runi thinks **maybe** fourteen" was intended (now

I think about it) to convey "even though Runi said fourteen, she wasn't really sure about it, and you shouldn't be either".

'***Think***' (usually 'I think') is, in some respects, a straightforward hedged performative (Lakoff, 1973, p. 490); it appears to be the most frequently-deployed hedge in my transcripts. For example, in this extract, Alex rejects my prompt to list ways of making twelve, and goes straight for a prediction. When I appear to question it, she affirms, hedges, then revises.

T3:43	Tim:	Any ideas about how many ways there would be say for twelve? For twelve you could have twelve ...
44	Alex:	[instantly] Six.
45	Tim:	Six ways?
46	Alex:	Yeah, I **think** so. Seven. Twelve add zero as well.

There is, however, a potential ambiguity (Stubbs, 1986) associated with this, and with other 'private' verbs such as 'believe', 'suppose' and so on. I would characterise the distinction as between epistemic and root meanings. Let me offer an example:

I:5.1 You could use calculus to find the minimum, but I **think** that completing the square would be more elegant.

The ambiguity here concerns whether the 'parenthetical' clause (Lyons, 1977, p. 738) 'I think' is being used to implicate:

- an uncertainty (epistemic meaning) concerning the validity of the substantive statement (whether or not completing the square would be more elegant); *or*
- a firmly-held position (root meaning) - that completing the square would indeed be more elegant - arrived at after consideration along with other tenable positions. That is, an assertion of what I judge to be the case.

In speech, the intended force may be made more evident by the location of stress in the utterance i.e. "I *think* that ..." for the epistemic meaning as opposed to "*I* think that ..." for the root.

The extracts which follow are from my interview with Anthony and Sam. In almost every case the stress is of the first (epistemic) kind.

T7:5	Tim:	... What I want you to do is to talk to each other and come to an agreement about how many different ways you can do it. OK? [...] And I'll just listen for a moment. How many different ways can you do that?

6 Anthony: Er, let's have a **think** ... Halves, um ...

[end of first extract]

T7:43 Tim: Eleven. Is that all the ways do *you [root stress]* **think**, or are there any more? [long pause] What do you **think** Sam? Do you **think** there's any more ways, or do you **think** that's all the possible ways of doing ...

44 Sam: There's more.

45 Tim: You **think** there's more? OK. What would, give me an idea of what another one might be, or what it might look like.

46 Anthony: What about ... you can put quarters into ten parts and like that can't you.

47 Tim: Mm ...

48 Anthony: Well if we put them in about nine parts ... if it, all the way, keep doing that, you might end up to number ten.

49 Tim: Are, right, um ...

50 Anthony: If you started at one.

51 Tim: Yeah.

52 Anthony: And I got a bit less, bit less, bit less, bit less, about um, slowly get to number two. Keeps going round. Yeah? Would that work?

53 Tim: Right, I **think** I get the idea. I mean can you sort of get us started, and we'll try and **think** it through together. [pause] Mm hm?

54 Anthony: Let's have a **think**. [pause]

55 Tim: Did you say we were dividing it up into ten?

56 Anthony: Yeah.

57 Tim: Right. So, what could one number be?

58 Anthony: You could have it into eighths as well.

59 Tim: Uh huh.

60 Anthony: I've heard of eighths, um ...

The interview with these two boys was unique in failing to develop a scenario for combinatorial generalisation. This was chiefly because Anthony proposed using fractions at a very early stage, and he then seemed driven by some irresistible internal force to consider denominators outside the scope of his competence. Matters were not helped by the fact that Anthony had been diagnosed as having an aphasic language disorder; over the years he had become expert in manipulating adults by diverting away from questions or topics which he was unable to understand or cope with. One could view such strategies as a lack of awareness of the cooperative principles, in particular of the maxim of Relevance. An alternative interpretation would be that Anthony is expert at flouting that maxim for his own ends. Since Anthony was more assertive than Sam, and I wanted to allow the children to have a part in shaping the agenda (my contingent questioning), the discussion proceeded at times like a script from Monty Python's Flying Circus.

In T7:6 and T7:54 Anthony is 'simply' stating his intention to engage with a problem [T7:6] or task [T7:54]. Actually, on the evidence of the data, this is not at all typically childlike (to announce an intention to think). It is very much the mark of the confident adult who 'formulates' a conversation; that is to say, who describes some feature of aspect of the conversation within the conversation itself (Garfinkel and Sacks, 1970). It is the response of the person - a lecturer, for example, holding forth within their specialist field - to a question for which s/he does not have an instant answer. The announcement "I shall have to think about that" has the effect of:

- flattering the one who asked the question: it suggests that the question is non-trivial, so that 'even I', the expert, will need to pause for thought before I answer;
- making space for the speaker to arrive at a response, either by recalling some information that is not at the forefront of her/his memory, or (more impressively) by the exercise of reason upon available information. I repeat, because it is important, that in either case the thinking is, as it were, on display for public observation, or even public entertainment;
- implicating uncertainty without undue discomfort. It is not uncommon for the presenter of a mathematics lecture (or lesson at any level) to leave unfinished the detail of parts of calculations or arguments in her/his prepared notes so that students may be sure to witness a public (if somewhat contrived) resolution of uncertainty. [Note 5.5]

With Anthony, however, what came across to me (and was later supported by information gleaned about his social strategies for coping with aphasia) was a plausible and well-used device for mimicking cooperative intellectual effort, with the intention of retaining, or even gaining, the teacher's goodwill. Note that my first 'I think' in T7:53 is epistemic, whereas the second is a proposal that we do some thinking. I asked Anthony's class teacher what she made of his "let's have a think". She pointed out that successive teachers, concerned about his apparently erratic train of thought (and the psychologist's suggestion that he experiences difficulty in connecting ideas to form coherent thought-sequences), would repeatedly have offered him advice using a formula such as "Now think about it first, Anthony". Add to this the fact that this is mathematics and I am perceived as the teacher. The close, almost individual, attention I was giving to Anthony would be very familiar to him, as he was accustomed to having special, one-to-one support in recognition of his learning needs. It is likely that Anthony construes my questions as being of the 'testing' kind (Ainley,1988), whilst I intend them to function as what Ainley calls 'directing questions', to provoke the boys into thought about a problem. I shall not develop further here this mismatch of perception, but the issue is clearly one of very general relevance and significance. Anthony is likely to be skilled in eliciting clues from teachers as he navigates his way through the fog towards a response to their testing questions, to which they already know the answers. Anthony wants me to feed him some more clues. For example:

T7:52 Anthony: And I got a bit less, bit less, bit less, bit less, about um, slowly get to number two. Keeps going round. Yeah? Would that work?

The problem is that (a) I was unaware, at the time of the interview, of his particular manipulative skills, and (b) he doesn't know I'm playing a interviewer's game called 'contingent questioning'! Far from pulling him back on the rails when he wanders off, I tag along with him. Anthony seems to be manipulating the situation in order to delay genuine intellectual engagement with the problem posed.

This throws up, in an urgent but rather unexpected way, a fundamental methodological issue, which is this: that there seems to be a problem with contingent questioning as a research strategy in the (hopefully unusual) circumstances described involving Anthony. Indeed, the problem may not be peculiar to research on human cognition by means of contingent questioning, although the outcome can be bizarre in that case. The methodology assumes that the response of the 'subject' to the researcher in the interview is authentic i.e. genuine, sincere and cooperative. In other words, the discourse is interpreted on the assumption that the spirit of the Grice's cooperative principle and maxims is being respected. If the subject is not constrained by cooperative norms, the assumption is false and the conclusions liable to be suspect.

'about', 'around'

Channell observes that 'about' and 'around' appear to be interchangeable Approximators, and that the first is more common in speech. I shall examine here their use by three different children, in two extracts from the data. The first is with Harry and Alan.

T4:39	Tim:	So how many ways is it Alan?
40	Alan:	Nine.
41	Tim:	Nine, right. [pause] What if instead of saying two numbers adding up to ten I said two numbers adding up to twenty?
42	Harry:	That would be **about**, yeah I think ... that would be eighteen
43	Alan:	[simultaneous] ... Eighteen ways. **About** eighteen probably.

The second, from Frances/Ishka Episode B, includes use of 'around'. In fact, the pragmatic analysis of 'about' which follows could be applied equally well to 'around', and be illustrated from this extract and elsewhere in the corpus.

T6:129	Frances:	Fifty?
130	Ishka:	**About** fifty yeah.
131	Tim:	**About** fifty. Now are you saying **about** fifty, Ishka, because you're sort of playing safe or I mean do you really think it is fifty?
132	Ishka:	Well maybe not exactly, but it's **around** fifty basically? [...]
134	Frances:	Maybe **around** fifty.

In each case a prediction is being made - the number of ways of making 20 [T4:42, T4:43] and 100 [T6:130] - and each time the hedge is an Approximator (a Rounder, in fact) at the surface level. Channell (1994, p. 46) has found that respondents typically understand 'about *n*' to designate a range of possibilities, symmetrical about the exemplar number *n*. So the boys predict that the number of ways to make 20 is in the region of eighteen, maybe more, maybe less. I suggest, however, that the deep level purpose and function of the hedge is Shielding against possible error in the cognitive basis of their prediction. This suggestion is supported by closer inspection of the data in context. Harry and Alan have already listed ways of making 10, and decided on nine positive integer possibilities, allowing reversals but not including zero as a summand. On being presented with the second problem (making 20) it was more common for children to list and count again, as Frances and Ishka do in Episode A.

Harry, however, is a confident boy. [Note 5.6] He is a risk-taker, and goes straight for a prediction for making 20, avoiding the tedium of listing and counting. The basis of Harry's prediction seems to be proportional reasoning (doubling) - there are 9 ways for making 10, so there are 18 for 20. For fuller insight into Harry's thinking, his next contribution (following T4:43 above) is:

T4:44	Harry:	No I think nineteen.
45	Tim:	Eighteen, nineteen?
46	Harry:	I should write that again. [laughs]
47	Alan:	What's that?
48	Harry:	Up to twenty.

[Harry begins a list 10+10, 9+11, 8+12]

Later, and before the list is complete, he ventures

76 Harry: I think that'll be nineteen.

From the outset, then, Harry is uncertain as to whether the 'answer' to my question is 18 or 19. We just don't know how he arrives at these two possibilities. If his prediction is an extension of his experience of making ten, then (as already noted) doubling would produce Harry's first prediction. A more detailed awareness of the nature of his list of ways of making ten (which I tried to prompt in the later episodes of some Make Ten interviews) could have led to the second prediction. The fact that he articulates it ("No, I think nineteen") is all the more remarkable because his first, incorrect prediction is confirmed by Alan, albeit with something less than total commitment [T4:43]. It seems, then, that Harry may be entertaining these two different predictions from the moment I ask about making twenty, and he seems [T4:42] to be testing out the first possibility, not just for my consideration (and possibly Alan's) but also (perhaps especially) for his own:

T4:42 Harry: That would be **about**, yeah I think ... that would be eighteen.

The effect of the initial hedging is to allow himself some space for further consideration, and to declare uncertainty in the assertion which completes the sentence. In the end he resorts to listing and counting, presumably since he lacks sufficient confidence in either of his predictions to choose between them when I ask him to do so ("Eighteen, nineteen?").

My conclusion is that the hedge 'about', although classified as an Approximator, is being used by Harry in T4:42 principally to assist the communication of his propositional attitude; in particular, to serve Shield-like ends. Harry's attitude to his prediction, and my interpretation of it, is further reinforced by his use of the prototypic Shield 'I think'.

Ishka implicates the same attitude with 'about' in T6:130. My next turn in that conversation is evidence that I (as interviewer) suspected this intention:

T6:131 Tim: **About** fifty. Now are you saying **about** fifty, Ishka, because you're sort of playing safe or I mean do you really think it is fifty?

What I infer from Ishka's 'about' [T6:130] is not that she has approximated the actual number of ways to the 'round' number 50, rather that she is in possession of a generalisation, a conjecture which would lead to exactly 50 as prediction. Incidentally, it is normal practice to use round numbers as vague numerical reference points (Channell, 1994, pp. 78 ff.); indeed, a round number on its own may serve as a rounder (i.e. without a prefix like 'about' or 'approximately'), as in, for instance

I:5.2 A suit like that would cost you £300.

The fact that round numbers are normally chosen with numerical Rounders [Note 5.7] is further evidence in support of my suggestion that Harry and Alan [T4:42, T4:43] are deploying 'about' as a Shield, and not as a Rounder. If their intention had been to approximate rather than to hedge commitment, then Channell's findings would led me to expect 'about twenty' rather than 'about eighteen'.

My spoken contribution, then, in T6:131 is designed to test out Ishka's commitment to 50, asking "do you really think it is". Again, my use of 'think' here is in the root sense of 'believe', and I strengthen the probe by the adverbial Adaptor hedge 'really'. The intended effect is to encourage her to make her position "less fuzzy", as Lakoff puts it. Ishka's reply indicates her discomfort; she skilfully sidesteps my demand for commitment with a reply [T6:132] which amounts to a virtuoso performance in hedging. Even Frances, who at that stage is displaying more confidence than Ishka (and less hedging), double-hedges her response [T6:134].

'basically'

This is an interesting and relatively unusual hedge, used by only 3 of the 21 children, and only on this one occasion [132] by Ishka. [Note 5.8] The word can function as a 'bottom line' underpinning, synonymous with 'fundamentally, as in

I:5.3 John's problem is that he is basically lazy.

It seems to have the effect, as used by the children, of qualifying the content of what is being said or claimed; thus, it acts as an Approximator. The following extract is from an earlier, weakly-framed conversation with Simon (aged 12¾), which turned out to be a forerunner of the 'Make Ten' Task. Simon rapidly moved on from positive integer pairs to decimals. After a while, I intervened:

Si1:15	Tim:	What if I gave you one of the numbers, one point three recurring, what would the other number be?
16	Simon:	Em, eight point six recurring.
17	Tim:	Why?
18	Simon:	Because one point three recurring is **basically** a third ...
19	Tim:	You mean the point three ...
20	Simon:	... point three recurring is **basically** a third, so you need ... well, the one, that's one, so to make it up to nine you add on eight, then you need another two thirds, which is point six recurring.
21	Tim:	If you have, um, point three and point six recurring, and you add them up, what do you get?
22	Simon:	Point nine recurring. Mmm - nearly one.
23	Tim:	Nearly one.
24	Simon:	Yes.
25	Tim:	Why nearly one?
26	Simon:	Because it's not, because point three isn't, it's just nearly a third. It doesn't quite get to the third.
27	Tim:	When it's point three recurring.
28	Simon:	Yeh.
29	Tim:	Oh, so point three recurring isn't really a third at all?
30	Simon:	Well. it's very nearly a third.
31	Tim:	Very nearly a third.
32	Simon:	Yeh.

Simon's statement [Si1:18, Si1:20] that "point three recurring is basically a third" is not in fact an assertion of a fundamental (basic, so to speak) property of point three recurring. The adverb 'basically' is being deployed as a hedge, a Rounder in fact, so that the force of the statement is much the same as that of "point three recurring is approximately a third", or perhaps "point three recurring is as good as a third", much as one would say "97% is as good as full marks". It is as near as makes no difference. Brown and Levinson include 'basically' in a list of a dozen 'Quality hedges' (1987, p. 167), most but not all of which are archetypal Rounders ('approximately', 'roughly'), which "give notice that not as much or not as precise information is provided as might be expected".

Trace now the course of the above exchanges as the force of Simon's 'basically' is revealed in the questioning. My analysis goes like this: as it stands, Si1:18 is 'incorrect' - not that the true/false dichotomy is very meaningful when applied to hedged assertions (Lakoff, 1972) - although the intention is clear to me. In Si1:20, Simon responds to my prompt to correct, or perhaps to clarify his statement in Si1:18. In fact, he interrupts my prompt to self-correct and (re)states that "point three recurring is basically a third". On the other hand, he completes the arithmetic in Si1:20 with "you need another two thirds, which is point six recurring". Notice that there is no 'basically' this time. I (in my role as interviewer) am aware that confusion about the value of infinite decimals is commonplace with students - of all ages. This is not intended to be a patronising remark, given the range of foundational (basic, even) issues which underpin any position on the matter (Cornu, 1991). The issue here is the usual psychological and notational difficulties associated with equating an infinite series (the decimal) with its sum (the fraction). My strategy, in order to ascertain where Simon stands in relation to these two recurring decimals - determined 'on the hoof' as the "um" [Si1:21] indicates - is to ask him about their sum. As I expect, his reply conveys his belief that the sum falls short of one. Asked to explain, Simon is guarded but more explicit:

Si1:26 Because point three isn't, it's just nearly a third. It doesn't quite get to the third.

I press the conclusion in Si1:29, the "Oh" attempting to convey some neutrality, some surprise, so as not to put words into his mouth. But he remains uncertain, and unable to agree without qualification to the bald statement that "point three recurring isn't really a third at all". His reluctance to concede is marked by the maxim hedge 'Well,' [Si1:30] as he flouts the maxim of Manner, and arguably others besides! The whole exchange is marked by Simon's desire to be cooperative, yet true to himself, his beliefs, and his uncertainties.

'WELL': MAXIM HEDGES AND DISCOURSE MARKERS

Carlson (1984, p. 37-38) analyses the occurrence of 'well' in dialogue in terms (*inter alia*) of failure to meet the demands of a question. Speakers tend to preface answers with a discourse particle such as 'well', to indicate some sort of insufficiency in the answer to be given (Lakoff, 1973). Examples given by Carlson (who is content to offer literature as pragmatic data) include the following from Agatha Christie (1977, p. 22)

> Reflect a minute, Hastings. One can catch a murderer, yes. But how does one proceed to stop a murder? - Well, you, - you - well, I mean - if you knew beforehand - I paused rather feebly - for suddenly I saw the difficulties.

The following fragment is extracted from Appendix 2:

S0:1	Tim:	How many three-quarters are there in a hundred?
2	Simon:	**Well**, there are seven three-quarters in ten, remainder a quarter.

In effect, 'well' acts as 'maxim hedge' in such instances (Brockway, 1981) - the speaker is serving notice to the hearer that the contribution about to come will in some respect fall short of the requirements of one or more of Grice's maxims. In these examples, adherence to the maxims of Manner (Christie) and Quantity (S0:2) are in question. Perera (1990, p. 217-222) found that 'well' occurs the most frequently of eight "characteristically oral" constructions that she examined in the Fawcett corpus (Fawcett and Perkins, 1980) but she offers no pragmatic account for this observation. There is, however, quite a substantial body of literature on such particles (for example Wierzbicka, 1976; Carlson, 1984; Schiffrin, 1987; in addition to the work of Lakoff and Brockway); I am bound to draw on and illustrate it only sparingly and partially here.

Wierzbicka analyses 'well' as a 'pragmatic particle', a word whose function is to express a pragmatic meaning at minimal cost. She uses the term 'pragmatic meaning' to refer to factors of propositional attitude such as assumptions, attitudes and intentions. Considered against the backdrop of Grice's maxims, 'well' can frequently be argued to attach some vagueness to the speaker's compliance with one or more of the maxims. Indeed, this device is not uncommon in my Make Ten data:

T10:181	Tim:	Thirty-nine ... why, how do you know that?
182	Susan:	**Well**, you've got the, you've got your, let me see, nineteen ways, and then you've got another set of nineteen ways going the other way.

It could be argued that Susan can foresee rather a rambling account ahead, likely to violate the maxim of Manner. 'Self repairs' (false, starts, self-corrections) abound in T10:182. This is commonplace (in the transcript data) when people are asked to supply a reason for a belief, or an explanation of some sort. In the example below, I am talking with Lucy and Rachel: I have introduced the conversation with my usual gambit, and whilst Lucy sets about listing ways of 'making' 10, Rachel quietly indicates that there will be ten ways.

T2:14	Tim:	So you're saying, Rachel, even before Lucy's written them all down, you're saying that there'll be ten ways.
15	Rachel:	Mm [in gentle confirmation].
16	Tim:	How did you know that, before Lucy had written them all down?
17	Rachel:	Well, because if you've got a number that adds up to ten, the, em, there's ten, and you've got all the others down below. You can only make ten ways to get up to ten.
18	Tim:	Do you understand that, Lucy? [Tim doesn't. Lucy nods]. I don't, you explain it to me.
19	Lucy:	Em ... well ... there's only ten ways to make ten.
20	Tim:	Well, I can see you've only, I mean you've written down ten ways, right, if you count them up there's one, two, three ...
21	Lucy:	... and they're all different ...

In T2:16 I ask Rachel for an explanation: how did she know there would be ten ways before they were listed? The 'well' with which Rachel begins her explanation gives notice that I shouldn't expect an account which is entirely clear or convincing. Next Lucy is put on the rack, asked to clarify Rachel's argument. In fact she is only able to restate the conclusion, and her 'well' is encased in hesitation.

Brown and Levinson (1987, pp. 164-171) describe how some occurrences of hedges themselves (as opposed to discourse particles such as 'well', 'after all', 'anyway') may be interpreted as acting as maxim hedges. These hedges may do one of the following:

- emphasise that one or more maxim requirements are being met;

  T2:193 Roksana: **I do believe** there are thirty-eight now. [maxim of Quality]

- serve notice that (or indicate the possibility that) one or more maxim requirements are being flouted.

T4:99	Tim:	You think it might be seventy-two, Harry?
100	Harry:	It's a wild guess, but I ... I'm not sure about that ... I think it will probably be an even number ... most likely about that, because ... yeah, that is quite likely.

Harry's highly equivocal response flouts the maxim of Manner. The hedges simultaneously achieve that effect and point to it.

A CA perspective on 'well': Adjacency Pairs

An alternative, or complementary, way of understanding use of 'well' is in terms of preference organisation with respect to adjacency pairs. In Chapter 3, I described how dispreferred second turns are marked in various ways, including delays and prefaces. The use of 'well' to preface a reply to a request for information in the classroom can certainly be interpreted in this way. As Levinson (1983, p. 334) remarks, "The particle 'well' standardly prefaces and marks dispreferreds". Some uses of 'well' in the 'Make Ten' corpus (and elsewhere in my data) can be seen to be of this kind. In the extract above with Lucy, my first part [T2:18] is a invitation (to explain); Lucy's second part T2:19 amounts to a refusal, or at least an inability, to accept the invitation.

My response T6:20 indirectly evaluates Lucy's 'explanation'; in this case the indirectness marks my redressive action in anticipation of a Face Threatening Act.

For further evidence of the importance for pupils of 'well' as a pragmatic particle, look back to the Frances/Ishka Episode B [T6:118], also see Simon's utterance Si1:30 towards the end of the previous section.

HEDGES: THE TAXONOMY REVISITED

The account of hedges in mathematical discourse in this chapter is essentially that in my article (Rowland, 1995b) in which I accept without question the hedge taxonomy of Ellen Prince and her collaborators. However, I should like to offer here a slightly sharper [pragmatic, modal-oriented] cognitive and taxonomic view which, I believe, will provide a good conceptual basis for future discussion - in this thesis and elsewhere.

Consider two main categories identified by Prince *et al.*: Shields and Approximators. These are illustrated by:

I:5.4 I think there are ten beans in the jar [Plausibility Shield]

I:5.5 There are about ten beans in the jar [Rounder-Approximator]

In each case, a hedge fuzzifies the sentence:

I:5.6 There are ten beans in the jar.

The statement I:5.5 is arguably true if the number of beans in the jar is in fact 11, whereas I:5.6 is not. [Note 5.9] These Approximators have truth-conditional semantic consequences in the way that they modify a sentence. One view (Sadock, 1977, p. 434) is that Approximators not only alter the conditions under which a statement is true, but, by virtue of their vagueness, they "trivialise" its semantics and so render it "almost unfalsifiable". (Note Sadock's own use of an Adaptor, making his *own* claim almost unfalsifiable.) This is a semantic observation. A pragmatic perspective on the same claim, taking into account goals and intentions, ought to consider whether the speaker intended it to have that effect, and if so, why.

In contrast to Approximators, Prince *et al.* claim that "Shields [...] do not affect the truth conditions of the propositions associated with them" (*op cit.*, p. 89). I suggest that, in fact, the function of a Shield is to transform a proposition into a non-propositional speech act. The syntactic means of doing this is the use of a hedged performative ('I think'), an adverbial preface ('probably', 'apparently') or some other fuzzy meta-linguistic device (see Stubbs, 1986 for others). The illocutionary force of such constructions is that what was an (unhedged) statement sheds its status as a proposition (subject to truth-conditional semantics) to become a conjecture. The crucial effect is that the speaker has less stake (or none) in the truth or falsity of the (unhedged) statement.

For example, suppose I say:

I:5.4 I think there are ten beans in the jar.

I then carefully count the beans in the jar: there are indeed ten. This would be sufficient (irrespective of what I might believe about the number of beans in the jar) to render the statement I:5.6 true; but it does not render I:5.4 either true or false, since the conditions under which I:5.4 are true are strictly independent of the number of beans in the jar. They have to do with my beliefs, my propositional attitude, my state of mind. The pragmatic effect of I:5.4 is to implicate that the speaker doesn't know exactly how many beans there are (since otherwise s/he would be violating the maxim of Quantity); that s/he entertains the possibility that there are ten; but that they are unwilling to be held to be committed to the truth of such an assertion.

Consider once more:

I:5.5 There are about ten beans in the jar.

Now Approximators are deployed (at times) precisely for the purpose of constructing a scenario within a proposition which is (almost) unfalsifiable. Why should a speaker want do that? One reason would be that s/he is uncertain and does not wish to be seen to be committed to a straightforward proposition for fear of being seen to be wrong. As with the Shield, the illocutionary force of the statement is conjectural. The suggestion of Prince *et al.* that (p. 95) "Rounders do not reflect any uncertainty or fuzziness but are rather a shorthand device when exact figures are not relevant or available" is somewhat hasty; indeed, it is ultimately untenable.

There are, in effect, two parallel taxonomies of hedges:

A: Into lexical categories as specified by Prince *et al.* Thus, 'about' is (i.e. has the typical form of) a Rounder, and so on.

B: Into the following two semantic categories, in parallel with that of modal verbs (as discussed in Chapter 3; see also Coates, 1983, pp. 18-22):

Epistemic hedges- conveying a state of mind such as lack of knowledge, (un)certainty, (lack of) conviction, commitment, etc.

Root hedges (after Coates, 1983, see below) - non-epistemic: for example, giving an appropriate degree of precision. A variety of pragmatic purposes (such as those listed in Channell, 1994) may motivate such a hedge, but displacement is not one of them.

The essence of my argument in this section is exemplified by what I see as the two different pragmatic meanings of the Approximator 'about'. The first is epistemic, and occurs in my data in situations requiring prediction or generalisation, such as

T4:43 Alan: Eighteen ways. **About** eighteen probably.

The pragmatic goal of 'about' is to implicate uncertainty, and to achieve protection against accusation of error by rendering the utterance unfalsifiable.

The second is root, normally (but not necessarily) associated with estimation. In this dialogue, C is a confident 10-year-old girl, I an adult interviewer.

M222:1	I:	Can you tell me how many sweets there are on the plate?
2	C:	[2 seconds] **About** twenty?
3	I:	Now, can you tell me how many sweets there are in the glass?
4	C:	[1 second] Ten.
5	I:	And do you think there are exactly ten?
6	C:	No! [laughs] **not exactly.**

No counting was involved. The answers [2 and 4] are rapid estimates; her amusement in [6] makes this clear. The Approximator 'about' is implicit in [4], ten being a 'round' number (Channell, 1994, pp. 87-89). Here, the pragmatic goal is to meet cooperative requirements to do with Quality and Quantity - not to make claims in excess of one's actual knowledge, and to judge how much detail is required in a given situation.

These matters will be considered more fully in the next chapter. The burden of much of this one has been that, in pupils' mathematics talk associated with predictions and generalisations, the semantic function of Approximators is usually epistemic. Similarly (but not so frequently), the semantic function of a Shield may be root, e.g. the 'private' verbs 'think', 'believe', and so on (see the discussion earlier in this chapter).

The epistemic/root approach encourages attention to the function of hedges irrespective of their form: it will also facilitate discussion in Chapter 6 of properties of hedges that have a great deal in common with those of modal verbs.

HEDGES AND POLITENESS

The use of hedges can often be seen as a means of redressing threats to 'face'. Consider, for example, the utterance:

I:5.7 I'm sort of hoping to get it finished by Friday.

in which the speaker's commitment actually to finishing by Friday is loose, and there is only the weakest sense of any kind of promise (a threat to negative face). This is very typical of use of epistemic hedges in 'Make Ten'.

T9:149 Rebecca: ... maybe it would be twelve because it's um ... half way inbetween.

The epistemic modal 'would' and the epistemic hedge 'maybe' redress the potential face threat to Rebecca (in case it turns out not to be twelve).

This is broadly consistent with my observation that I typically use hedges (Shields and Adaptors) in recognition of the face wants of the children, whereas they typically use Rounders and Plausibility Shields as epistemic hedges which render their conjectures almost unfalsifiable, in order to serve their own face wants.

Students who, in their own perception, enjoy a more balanced power relationship with their tutor do in fact have a concern for his or her face needs. This was apparent in a small way with the ten- and eleven-year-olds when they felt they might be usurping my role as 'teacher', and exhibited negative politeness. Here, for example, Caroline would like to explain something:

T3:53	Caroline:	Um, can I ...?
54	Tim:	Oh yes, please Caroline.
55	Caroline:	Half of ten is five and we actually got five, but then we added on ten add zero, so it would be six. So, so far it's basically worked out half the number, and it's the same with the twenty, it's eleven because we've added on zero and twenty, and it's the same with the sixteen.

Caroline adopts the face-redress strategy of posing her offer in the form of a question, before giving an extended explanation. Note the relative absence of hedging in [55] - here she is not coming to know the matter she articulates; rather, she *knows* it.

THE ZONE OF CONJECTURAL NEUTRALITY

In this chapter I have shown how children use Rounders and Plausibility Shields to implicate uncertainty, to insert some space between conviction and asserting a proposition. I suggest that that space, between what we believe and what we are willing to assert, deserves a name: I propose the 'Zone of Conjectural Neutrality' (ZCN). [Note 5.10] Even Rounders, such as 'about', which syntactically attach some fuzziness to the proposition itself, are pragmatically deployed by the children to achieve Shield-like ends. This, and the forms of linguistic Shielding which I have discussed, have the effect of reifying the ZCN and thus distancing the speaker from the assertion that he or she makes. Whilst truth and falsity may be decided in the ZCN, a person may articulate a proposition without necessarily being committed to its truth. In such a cognitive and affective *milieu*, it is the proposition that is on trial, not the person. Whilst mathematical conjectures are formed as private, cognitive (perhaps inductive) acts, they are validated in public polemic of some kind. Moreover, the learner ideally participates in the discourse since, as Balacheff submits (1990, p. 259), children must take responsibility for the validity of their own solutions "in order to allow the construction of meaning". At the same time, a conjecture is not fixed and immutable, but modifiable. I am describing, of course, the quasi-empiricist approach to teaching and learning which I described in Chapter 2. I referred then to Dawson's (1991) account of a "fallibilistic way of teaching".

> A teacher who is functioning fallibilistically [...] establishes a classroom climate in which an atmosphere of guessing and testing prevails, where the guesses are subjected to severe testing *on a cognitive rather than an affective level* [...] where knowledge is treated as being provisional. Because of the provisional nature of knowledge, pupils are encouraged to confront the mathematics, their peer group and, where appropriate mathematically, even their teacher. (Dawson, 1991, p. 197, emphasis added)

Not only is uncertainty an intellectually tenable position, but the assertion of uncertainty draws the attention of the teacher to the existence of a ZCN, and thus opens up the possibility that s/he might provide for the student some cognitive 'scaffolding' (Wood *et al.*, 1976) to support, and perhaps transform that state. This seems to be what is happening to Harry here, in a final extract:

T4:128	Tim:	How do you know there are forty-nine Harry?
129	Harry:	Well **I am not completely certain actually**, but I would expect it because if you start off with fifty and you do forty-nine add one, forty-eight add one,

		but then you'd end up with one add forty-eight wouldn't you, so they always change ... [...]
133	Harry:	Forty-eight add two I mean.
134	Tim:	OK. And the last one in that list would be?
135	Harry:	One add forty-nine, so they'd all be ... [interrupted by Alan sneezing]
136	Tim:	How do you know that there's forty-nine different ways that you've listed? You started with forty-nine add one and you ended up with one add forty-nine. Now how do you know that there are forty-nine pairs in that list?
137	Harry:	Well there's fifty numbers, and you just, there's lots of ways because you just go forty-nine add one, forty-eight add two all the way down 'til you get to the one, but you can't do fifty add nought, so that will take away one which will make you with forty-nine. **I'm quite certain about that.**

I shall return to further consideration of the ZCN in the final chapter.

SUMMARY

In this chapter, I have shown that the classification of hedges (due to Prince *et al.*, 1982) into functional categories is relevant and useful in the analysis of my task-based mathematical conversations with children aged 9 to 12, where children are being encouraged to predict and generalise. I have noted that:

- I (as interviewer) exploit Attribution Shields and Adaptors, usually for teacher-like purposes; *whereas*
- the children typically use Rounders and Plausibility Shields, and nearly always to implicate uncertainty, to insert some space between conviction and asserting a proposition. *Furthermore,*
- I have proposed that the space between what we believe and what we are willing to assert be recognised, and that it be named the 'Zone of Conjectural Neutrality'.

The purposes which speakers achieve by the use of vague expressions (Channell, 1994, pp. 186-9) include "displacement" (in case of uncertainty) and "self-protection" (a safeguard against later being shown to be wrong). Given the prevailing school-culture (maths is about right and wrong answers, and it is much better to be right), the use of hedging is evidently deployed by many children as a Shield against being 'wrong'. These Shields could be seen to act as linguistic pointers to intellectual 'risks', with attendant vulnerability. In principle, it would be preferable for students to know that being unsure is a genuine, valuable and creative option available to them.

CHAPTER 6: ESTIMATION AND UNCERTAINTY

Tip number 448: Don't be afraid to say, "I don't know"

(H. Jackson Brown Jnr., 1991: *Life's Little Instruction Book.*)

From time to time a pupil may feel obliged to make an assertion, perhaps in answer to a question, yet without certainty that what s/he is claiming is entirely accurate or true. The 10- and 11-year old children considered in the previous chapter had available a repertoire of hedging strategies for maintaining cooperative interaction whilst being appropriately vague, thereby conveying a lack of full commitment to the propositional content of their utterances. Further evidence of this capability in other students will be presented in the next chapter. When is this linguistic repertoire developed, and are there identifiable milestones on the way to confident mastery?

It will be seen that the nature of the question to be addressed - the development of epistemic modal forms and hedges over the primary school years 4 to 11 - requires systematic collection of appropriate spoken language data from a representative set of children across that age range. The vague language studied so far in this thesis, with some two dozen children, arose in contexts where pupils were engaged in activities involving prediction and generalisation. This required extended, contingent interviews, in order to prepare the ground, i.e. the problem environment, for these mathematical processes to come into play.

Much of Channell's recent book (1994) on the pragmatics of vague language is concerned with approximating *quantities.* The study reported in this chapter focuses on that dimension of vague language, specifically on estimation of the number of objects in a set. This choice of focus is partly for the sake of addressing what is perhaps the most obvious aspect of mathematical activity in which one would expect vague language to play a part. Moreover, it is possible in a short (5-10 minute) interview to present appropriate estimation tasks to children in a meaningful way, to obtain responses, and to follow up from a restricted menu of probes. It is therefore convenient, in designing an age-related study, to use estimation rather than generalisation tasks to elicit vague language when dealing with a pupil sample numbered in hundreds rather than tens.

ESTIMATION

In his recent thesis, Clayton (1992, p. 11) classifies the diffuse notion of estimation into three broad categories.

Computational estimation involves the determination of approximate (typically, mental) answers to arithmetic calculations e.g. 97π is roughly 100x3.1 or 310. Such competence is commended by the National Curriculum (DFE, 1995, p. 25) for the purpose of checking answers to precise calculations for their 'reasonableness'; pupils, however, seem to regard such checks as trivial or pointless (Clayton, *op cit.*, p. 163).

Quantitative estimation indicates the magnitude of some continuous physical measure such as the weight of a book, the length of a stick.

Numerical estimation entails a judgement of 'numerosity' - the number of objects in a collection. In principle, such a set could be precisely quantified by counting. In practice such a precise enumeration may be impracticable or simply judged to be unnecessary, excess to pragmatic requirement.

Ellis (1968, p. 159) observes that counting may be considered to be a measuring procedure, but is unique in the non-arbitrariness of the unit of measure. Nonetheless, Clayton merges numerical estimation and quantitative estimation into one analytical category. This obscures the fact that there is sparse reference to numerical estimation in his literature survey (pp. 23-42).

A surprising justification for this apparent omission emerges from a reading of Judy Sowder's review of research on estimation for the NCTM *Handbook* (1992). Sowder notes that "there simply is not a rich research base in estimation" (p. 372) and that most such research has been on computational estimation. Moreover, "Numerosity estimation has received the least research attention, and [...] the only two studies located combine it with measurement estimation" (p. 372). A close reading of her review suggests that one those two studies was reported in a short article - by Clayton himself - in *Mathematics Teaching* (Clayton, 1988). The numerosity component of the other (Siegel *et al.*, 1982) analysed estimation competence in terms of 'benchmarks' (known standards) and 'decomposition/recomposition' of a set in order to apply a benchmark together with a computation. A variety of estimation tasks (of quantity and numerosity) were presented to children aged between 7 and 14, and to a small sample of adults. The investigators found marked developmental differences in performance on numerosity items. Note that the number-estimation tasks (judging from the examples given in the paper) seemed to involve quite large sets. Example: "How many names on a page from the phone book?". The significance of set size will be considered later in this chapter.

The literature on estimation of quantities does indeed tend to be about estimation of measures. Nobody really knows why this portion of the physics curriculum (on measurement) has been appended to primary mathematics in the UK. It seems to come down to Edith Biggs, Nuffield Maths (1960s style), practical work (a Good Thing), and the fact that measurement is self-evidently 'practical'. Janet Ainley (1991), setting out to investigate the *mathematics* in measurement, describes a staple-diet "estimate then measure" lesson with eight-year-olds. Reflecting on the lesson, Ainley comments (p. 70) on the peculiarity of estimating and *then* measuring. Clayton (*op cit.*, p. 23, p. 158) independently agrees:

> Most estimation tasks in school require an estimate and then (almost immediately) a measure or calculation is made. Many colleagues have agreed with me when I have asserted that pupils often make their 'estimate' **after** they have measured or calculated showing their disregard for the estimation process. (p. 23).

The behaviour of the children (in Ainley's account) certainly reinforces the Ainley-Clayton observation; some of them enter the measurement of another child for their 'estimate' - it is, after all, so much more satisfactory if the two agree. On the other hand, measuring devices are calibrated discretely, whereas mass, length, time and all quantities derived from them are continuous. Thus, as Bright (1979, p. 581) observes:

> Every measurement is an approximation, or if you will, an estimate.

Incidentally, Ainley concludes that:

> There is mathematics in measurement; but it does not happen to be in the bits which currently get given priority in mathematics lessons. (p, 76)

The study to be presented in this chapter is concerned with children's estimates of numerosity, and their spontaneous production of vague language in articulating and discussing such estimates. Suppose, for example, that I ask a 'phone-book' question such as "How many words are there on this page?". You (the reader) are likely to respond - perhaps from experience of reading essays or writing papers - along the lines "About four hundred". The hedged approximation in such a response can be accounted for by reference to one or more of the pragmatic goals listed towards the end of Chapter 2.

The first of these goals (giving the right amount of information) accords with Grice's maxim of Quantity - "Let your contribution be informative but not too informative" - for cooperative interaction. The point is that, in saying or writing "About four hundred", you have judged that I don't much care whether there are actually 388 or 413 words on the page.

The fourth goal (covering for lack of specific information) could also account for the same hedged response. In this case, you actually don't know how many words there are, and can't be bothered to count them. At the same time, by saying "about 400" you do observe one of Grice's maxims of Quality - "Be truthful: don't say that for which you lack evidence".

The last goal in Channell's list (protecting oneself against making mistakes, against accusation of error) is equally pragmatically plausible. For one consequence of the vagueness of the response is this: it would be very difficult for me to demonstrate that the claim contained in it was wrong. The hedge is epistemic, and works for the speaker because it effectively renders the claim unfalsifiable.

MEASURING AND ESTIMATION IN SCHOOL

Prince *et al.* (1982) and Channell (1985;1990) have demonstrated how speakers and writers deploy hedges in order to fulfil a variety of goals. Their studies were mainly of adult academic and professional groups, such as doctors, copywriters, broadcasters, students of linguistics and economists. Both identify the prevalence of a protective purpose - the recognition of vulnerability and the consequent need to need to 'shield' oneself. I have already argued, in Chapter 5, that this need is ever-present in public settings, usually schools, where pupils do mathematics.

Ainley (1991, p. 70) says, *en passant*, of the estimation lesson she observed:

> It is a relaxed lesson: estimates are meant to be wrong, so no one is worried about failure.

I know exactly what she means; the lesson is relaxed in the sense that it demands little of the children. But I'm not sure that these (or other) children have been let in on the secret that estimates are meant to be wrong; in any case, 'expected' would be more fitting than 'meant'. If too many of a child's estimates agree with the 'right' (i.e. measured) answers, then the teacher suspects foul play - a classroom form of Tiegen's paradox (Chapter 2) in that the child who is most accurate is deemed to be the least likely to have arrived at his 'estimate' by fair means. This is a strange mathematics classroom game, in which 'right' estimates are more likely to meet with the teacher's disapproval than plausible 'wrong' ones. That this is not known to the children is evidenced by the way many of them subvert the activity - estimate *after* measuring, not before. It is a first step on the rung of a ladder of innocent deceit: nearer the top is the A-level science student who draws a straight line graph, then plots some points plausibly arranged either side of it, before finally tabulating his 'experimental' results.

Of *course* it matters to the student and his teacher if the data do not fit the theory: 'experimental error' tolerates only modest disturbances from the neat and tidy world of the model, the 'theory'. Notwithstanding a relaxed atmosphere in the classroom, I suggest that eight-year olds *are* worried about failure when they do estimate-and-measure in school. Not worried in a debilitating sort of way perhaps, but enough to want to fix the answers.

Weiner (1972) identified a vicious circle in children's attitudes to estimation: poor estimators, not surprisingly, viewed estimation as "risky", avoided it, and remained poor at estimation. In any case, it may not be at all clear to such children (from their experience or from any words of the teacher) that estimation is something that they can get better at by practice. Clayton develops this affective theme in his thesis *Estimation in Schools*, in which he presents and studies estimation as a risk-taking activity. Transcripts of pupils performing estimation tasks were analysed to determine a (somewhat subjective) measure on a scale 1 to 10, of pupil confidence in the estimates they gave, using indicators such as "willingness to explain methods, general air of confidence". The judgement of confidence was irrespective of the suitability or accuracy of the estimate. Clayton's conclusion is that the boys in his sample were generally more confident than the girls. He followed up with a questionnaire to pupils in primary and secondary schools to assess attitudes to familiar risk-taking situations e.g. volunteering to answer a question in class. For the secondary pupils, a consistent pattern emerged in which boys were judged more confident than girls in such situations. For primary pupils the data were not so consistent or significant in this regard, but Clayton nevertheless concludes (p. 149) that "gender plays a strong role in pupils' willingness to engage in activities in school that involve risk". I shall return to this issue towards the end of this chapter, and muddy the gender water a little more.

COUNTING

The earlier question about the number of words on a page tacitly invites an estimate of numerosity - the cardinality of a discrete, finite set which, if an exact answer were required, could be counted. One of the tasks presented to children aged 4 to 11 in the present study was designed to offer the choice of counting or estimating a set of 19 objects. Data on the methods used by those children who did opt to count was collected and coded in the course of the study. This counting data is a kind of by-product of the linguistic core of the data, but will turn out to have some relevance in interpreting the linguistic behaviour of these children who are presented with a mathematical task involving numerosity.

In the early years of schooling, children are explicitly and laboriously taught to count, but not to estimate, small finite sets. The process of counting has been minutely studied and analysed, notably in the USA, by Zaslavsky (1973), Gelman and Gallistel (1978), Steffe, von Glasersfeld, Richards and Cobb (1983), Steffe and Cobb (1988) and Fuson (1988, 1991). In essence, counting a finite set entails the matching of the elements of the set (in any order, but without repetition or omission) with a fixed set of words - number-names ('tags') - which must be produced for word-object matching in a canonical sequence. Thus, a set with n elements can be counted in $n!$ different ways. Gelman and Gallistel (1978, pp. 77-82) identify five organising principles in young children's counting:

- the stable order principle - the tags must be drawn from a stably-ordered list;
- the one-one principle - every item in a set must be assigned a unique tag;
- the cardinal principle - the last tag used is the cardinality of the set;
- the abstraction principle - the above principles can be applied to any collection;
- the order-irrelevance principle - the order of enumeration does not affect the outcome of the count.

Gelman and Gallistel conclude (p. 130) that the first three "how-to-count" principles are learned in that order; for example, a child can reliably recite the list of number-names before s/he can assign them injectively to a set of objects. Fuson questions the invariability of this learning sequence, finding that it depends on the size of the set to be counted. In particular, for sets of cardinality above 16, it is the one-one assignment that causes most difficulty, and (not surprisingly) this is especially the case when the set is disorganised rather than being presented in a row.

Fuson also draws attention to the fact that when a person performs a count, they have to find some way of coordinating the word-object correspondence, and that this is achieved by two simultaneous kinds of pointing 'actions', or "indicating acts". First, the person doing the counting has to point systematically to the *objects* in some (complete, non-redundant) order. Secondly, and simultaneously, they must point (in the sense of drawing attention, at the very least their own attention) to the number-*word* which is to be matched with the objects as they point to them in turn. Fuson calls these two independent indicating acts 'local correspondences'. Both must be one-one if the count is to succeed - "one word must correspond to one indicating act and one indicating act must correspond to one object" (Fuson, 1991, p. 31). The form of each of these indicating acts undergoes change with growing maturity, but invariably begins from an externalised paradigm - the earliest object-indicating act is achieved by touch: the word-indicating act is speech, specifically by counting aloud. The first is spatial-tactile, the second linguistic. It is as though nature had been careful to assign one task to each cerebral hemisphere. [Note 6.1] Between the ages of about 3 and 6, and beyond, each of the two pointing actions attenuate to internalised versions:

> Both action parts of counting immovable objects - pointing and saying number words - undergo progressive internalization with age. Pointing may move from touching to pointing near objects to pointing from a distance to pointing from a distance to using eye fixation. Saying number words moves from saying audible words to making readable lip movements to making abbreviated and unreadable lip movements to silent mental production of number words. (Fuson, 1988, pp. 85-6)

The whole gamut of these 'actions', the human repertoire of word-act-object associations, from touch-say to gaze-mute, was demonstrated in the data collected from these short interviews with 230 children aged 4 to 11.

Ginsburg and Russell (1981) report that moves towards internalisation of counting actions result initially in reduced accuracy. Nevertheless, Saxe and Kaplan (1981) found that this loss of accuracy is recovered: six-year-olds in their sample were as accurate in counting an array that required an internal indicating act as in counting one for which external pointing could be used. Moreover, Briars and Fuson (1979, unpublished raw data, cited in Fuson, 1988) found that the external-internal progression will be reversed according to the requirements of the counting task; high school students counting large disorganised arrays first counted without pointing (with eye fixation), but then either spontaneously changed to pointing or did so when the experimenter observed that it was a hard task and "you don't need to do it in your head".

Abutting and overlapping the teaching of counting (at age about seven), the *activity* of estimation - of lengths, weights, and so on - features strongly in the primary school curriculum, even though there may be little discussion of the process of estimation, and consequently little attention is given to developing techniques to improve pupil's competence to make 'good' estimates (Ainley, 1991, p. 73).

ENQUIRY FOCUS

It has been suggested [Note 6.2] that when a teacher (or textbook, workcard, etc.) asks "How many?", young children (in the first two or three years at school) typically receive the question *as an invitation to count* rather than to estimate. In the context of primary school mathematics, the suggestion is a very plausible one. Counting a small set may be regarded as a less risky enterprise than estimating its cardinality. If this is the case the young child's response to such a question is less likely to be modalised or hedged than the older child's. Furthermore, the young child will not be able to hedge until s/he has learned how to achieve that effect with language. My expectation, then, is that modal forms and hedges will be relatively absent in mathematics talk in early childhood, and that one can discern progressive development of modal/hedging capability and use in individuals through the years of primary schooling.

The aim of this enquiry was to examine the validity of this expectation, looking for trends in the responses of pupils across the 4 to 11 age-range, in the context of cardinal estimation activity.

I have divided both modal forms (Chapter 2) and hedges (Chapter 5) into two broad categories, labelled (in both cases) epistemic and root. The epistemic category contains instances of language use (modals or hedges) which encode and serve to convey the speaker's attitude to or confidence in what s/he is saying. The category is therefore pragmatically determined but, particularly for modals, there are semantic parallels i.e. what kind of modality (wish, conjecture, etc.) is it? The root category is, by definition, non-epistemic. A working test for a root modal or hedge might be that the speaker has little or no affective 'stake' in what s/he says. Thus. deontic modals (of obligation and permission) would normally be root, as would an Approximator-hedge that could be claimed to be motivated by concern to respect the maxim of Quantity.

In this study, however, both modals and hedges will be identified in the first instance by reference to their form rather than their pragmatic function. Hedges will be identified as Approximators or Rounders, and modal language will be identified entirely by the presence of modal auxiliaries. This is easily justified since modal verb moods and tenses (see Chapter 2) are marginal in modern English (Stephany, 1986, p. 385), especially in speech, and so the modalising function falls on modal verbs (auxiliaries). The adverbial modal forms such as 'possibly' and 'maybe' are in any case included as hedges (Shields).

It could be argued that every epistemic hedge is in fact a modal use of language. The case is already made for adverbial Shields. Stubbs (1986, p. 18-19) clearly takes the Shields 'I think/believe/guess/etc./ that' and 'it seems that' to be modals when they release speakers from total commitment to propositions (i.e. when, in my terms, they are epistemic hedges) as opposed to when they are used to make statements about what Stubbs calls "private psychological states" such as dogmatic conviction.

METHOD

The study was carried out in a 4-11 primary school. There were some 230 children on the school roll. Every child was asked the same three "How many?" questions in private, one-to-one interview. Details of the questions and related tasks are given below. The object was to test the expectation that the language of modality and hedging will be more commonplace among the oldest children (10-11) than the youngest (4-5), with some sort of continuum evident between these extremes.

The fieldwork was carried out by a student assistant (the "interviewer") in the last month of the school year. The interviewer was well-known to both teachers and children in the school. The tasks had been piloted in another school so as to train the interviewer and refine the precise wording of the questions themselves. In their final form these were as follows.

- Task 1: The interviewer produces a white plate on which 19 sweets have been placed so that each is visible. The sweets are similar in size and appearance to "Smarties". [Note 6.3] The child is asked, "Can you tell me how many sweets there are on the plate?". [Note 6.4]
- Task 2: The interviewer produces a high-quality colour photograph of a small glass containing 14 sweets. These almost reach the rim of the glass. The child is asked "Can you tell me how many sweets there are in the glass?"
- Task 3: The interviewer shows the child two thin plastic tubes (both are about 25 mm in diameter and 10 cm high). One contains 10 sweets, the other 20. The interviewer says "There are ten sweets in this tube (indicates). I know that, because I counted them when I put them in. Can you tell me how many sweets there are in this (indicates the other) tube?"

The materials used are shown together in the Plate overleaf, although the materials for each task were produced in turn by the interviewer as they were needed.

Why choose these particular three tasks? The point about the first is that the child can actually count the sweets if s/he chooses to do so, but may also make a reasonable estimate if s/he so chooses. The actual number is a determinate and accessible quantity; the child must decide whether it is required, and be aware that estimation is an option. Before I piloted this task, I had considered some numerical variation for the youngest (age 4-5) children, placing just 9, or even 6, sweets on the plate. At the same time, I had a preference for keeping the task constant across all the age groups, otherwise it would be possible to account for differential responses by the fact that they had been given different tasks. Early years teachers with whom I discussed this dilemma advised in favour of fewer sweets with the youngest children (having in mind, I suspect, the limitations of the few rather than the capabilities of the majority). I had to have good grounds for believing that the enumeration by counting of 19 sweets would be an accessible option for the four- and five-year-olds in the sample. Whether the count was *accurate* was, for the purpose of the study, immaterial. Gelman and Gallistel's study gives evidence that:

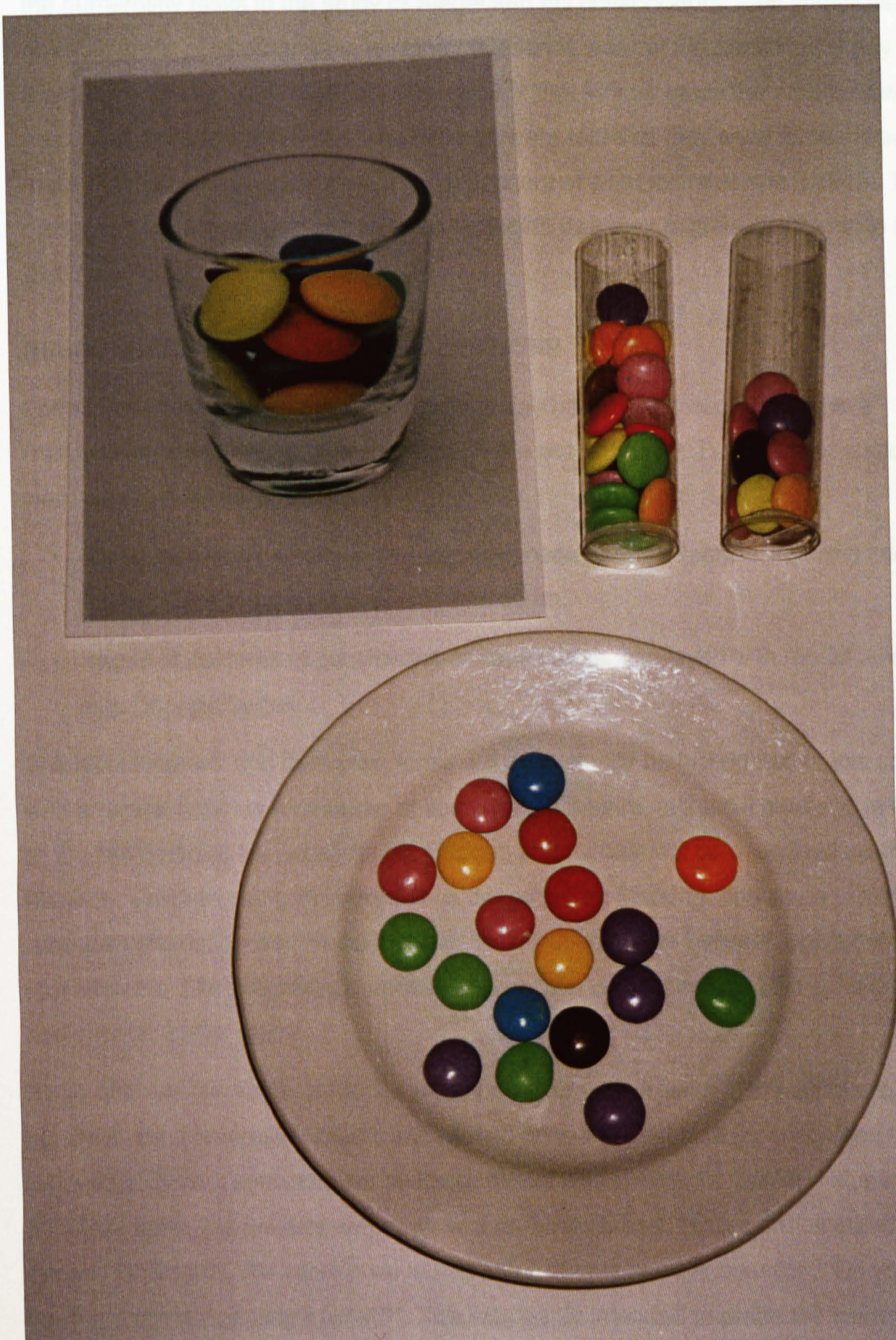
> 4- and 5-year-olds can [assign tags to items] for set sizes up to 19 [...] young [meaning 3-year-old] children do *not* treat set sizes in excess of 5 as undifferentiated *beaucoups*. (p. 111).

In the event, both the pilot and the survey-proper vindicated not giving the youngest children an 'easier' task. I was concerned, too, that if a child could 'see' how many sweets there were by direct perception - exact enumeration without the need to count is called 'subitising' in the literature (Jensen *et al.*, 1950) - then the issue of estimating would not arise as an option.

The second task was designed so that the precise number of sweets in the glass was indeterminate. It can not reliably be determined by counting, since not all of the sweets are visible in the photograph. Some kind of estimate is therefore necessary, and some degree of uncertainty is likely to be present in the situation, although the estimate may be guided by a count of the sweets visible in the photograph. Uncertainty may be lessened by naive interpretation of the two-dimensional image, i.e. failure to realise that some sweets which were present in the glass are not part of the photograph.

Similarly, in the third task, the precise number of sweets cannot be determined by counting, since not all are visible on the outside of the tube. A handful of the 230 children tipped out the contents of the tube and counted them that way! However, the height of the sweets in the second tube, relative to the first, is a possible guide to the number in it, given the fact that there are ten in the first. In this case, then, estimation may be guided by an elementary form of proportional reasoning, namely doubling (Hart, 1979. p. 99). An alternative perspective on this strategy would be to view it as 'regular decomposition/recomposition' (Siegel *et al.*, 1982, p. 213). The contents of the second tube are decomposed into samples, each estimated to be the size of the given 'benchmark' i.e. the contents of the first tube; the two samples are then recomposed and the answer computed. The interviewer sought to understand whether any such strategy and inference was a factor, using probes such as "How did you know that?".

PAGE MISSING IN ORIGINAL



Tasks 2 and 3 were essentially intended to "block" (Laborde, 1989, pp. 33-4) the possibility of complete solution by counting, in order to introduce an element of uncertainty and a need to estimate. All three tasks bore some superficial similarity to the numerosity tasks in the study of Siegel and his collaborators: their stimuli were all physical props or photographs, and their questions were of the form 'How many X's are there in/on this Y?'. The essential difference is that in their study the problems were presented *as* estimation tasks. The children were told that they were to be asked to make estimates, and were given a short account of what estimation is (p. 215). In the present study, it was up to the children to decide that they might estimate, or to infer that it would be necessary to do so.

RESPONSES AND CONTINGENT QUESTIONS

For all three tasks, each child was asked to say how many sweets there were (respectively on the plate, in the picture, in the second tube). Two kinds of response were categorised as '**Marked**':

- those responses which conveyed vagueness through specific linguistic hedges - 'I think there are ten', 'About ten', and so on;
- vague statements of possibilities or conjectures, conveyed with modal auxiliaries e.g. "it might be ten".

The label 'Marked' and derivative forms will consistently be highlighted in this chapter with a capital letter as a reminder of its current, if interim, technical meaning referring to the two itemised response-types. Hedges and modals will be described jointly as Markers. Children using Markers will be described as Marking, and so on. As already noted, no distinction was made in the data-collection phase between epistemic and root Markers. The pragmatic purposes of particular occurrences will be considered, however, at a later stage.

If one of these two kinds of Marker was spontaneously present in the initial response of the child, the interviewer noted it and moved on to the next task (or concluded the interview). Such a spontaneous hedge or modal was denoted a 'primary' Marker. If, on the other hand, the primary response was un-Marked (e.g. "There are nineteen" or simply "nineteen"), the interviewer would ask a supplementary question, "Do you think there are exactly nineteen (or *n*)?". This was partly intended to probe the child's commitment to their un-Marked answer; at the same time, my intent was to see whether the children who did not use Marked language spontaneously could be encouraged to do so, thereby revealing that they knew *how* to do so. If this second question provoked a Marker in the child's reply, then this secondary Marker was recorded. Thus, for each of the three tasks, primary and secondary Markers were mutually exclusive.

DATA

The interviewer prepared a response proforma (Figure 6.0) with a sheet for each child which he completed during the course of the interview. Every interview was audiotaped and a quarter were videotaped: the recorder was set up in advance of each session and just left running. Therefore, it was possible to return to the tapes, if necessary, to check the proformas and to study prosodic and other nuances of the children's responses. Along with hedges and modal auxiliaries, the interviewer recorded a number of other features of the response on the proforma. Most field-names on the proforma are self-explanatory. The 'Soundtrack' field contains (for many, but not all of the children) a digitised sound sample (copied from the audio or video recording) of Marked language.

The records of the children's responses, as entered on the proforma, were entered onto a database ('Data Power', Iota Software, 1994). The software enabled the usual data-interrogation methods. Most of the data relevant to this paper are shown in Table 6.1, which gives the number of occurrences of all Markers, separated into four age bands (see below). These same data are displayed in the bar charts in Figures 6.1 to 6.4. Occasional reference will be made to Table 6.2, which separates the same Marker data into finer categories, corresponding to three ability groupings (to be explained later) for each age-band.

Before proceeding to identify some trends apparent from the graphs, I shall comment on the rationale for the organisation of the data in the Tables and Figures.

Compulsory education in England and Wales is organised in chronological 'Years', normally beginning (in the absence of Nursery classes) at age four or five with between one and three terms in Year R (for 'reception'). The youngest children in Year 1 will be just five at the beginning of the academic year, the oldest nearly seven at the end. The 'Primary' phase of schooling covers Years R to 6.

In the Primary school which participated in the study, all children have three terms in Year R. The number of children in each school 'year' (and present for the interview) varied from 23 (Year 5) to 45 (Year R). The results on Marking are presented here in four year-bands rather than seven individual years. The bands are:

St Luke's C.P. School   Research: June 1994   Ref. No:

Class: *Gender:*   *Year:*

Age: *DecimAge:*   *Ability Level:*   *Band:*

Question 1: Can you tell me how many sweets there are on the plate?

Count: *Number:*   *Soundtrack:*

Aloud: *Rising Intonation:*

Hesitation: *Response Length:*   *Modal*

Primary Hedge:

Exactly:

Secondary Hedge:

Question 2: Can you tell me how many sweets there are in the glass?

Count: *Number:*   *Soundtrack:*

Aloud: *Rising Intonation:*

Hesitation: *Response Length:*   *Modal*

Primary Hedge:

Exactly:

Secondary Hedge:

Question 3: There are 10 sweets in this tube.
Can you tell me how many sweets there are in this tube?

Count: *Number:*   *Soundtrack:*

Aloud: *Rising Intonation:*

Hesitation: *Response Length:*   *Modal*

Comparison of Tubes:

Primary Hedge:

Exactly:

Secondary Hedge:

Figure 6.0

Table 6.1: MARKERS ACROSS 7 SCHOOL YEARS IN 4 BANDS							
TASK 1		Primary Markers - Task 1		Secondary Markers - Task 1		All Markers - Task 1	
Year	No.	No.	% of Year	No.	% of Year	No.	% of Year
R	45	4	9%	1	2%	5	11%
1 and 2	70	1	1%	4	6%	5	7%
3 and 4	65	1	2%	7	11%	8	12%
5 and 6	50	3	6%	10	20%	13	26%
TASK 2		Primary Markers - Task 2		Secondary Markers - Task 2		All Markers - Task 2	
Year	No.	No.	% of Year	No.	% of Year	No.	% of Year
R	45	4	9%	1	2%	5	11%
1 and 2	70	1	1%	4	6%	5	7%
3 and 4	65	6	9%	12	18%	18	28%
5 and 6	50	14	28%	10	20%	24	48%
TASK 3		Primary Markers - Task 3		Secondary Markers - Task 3		All Markers - Task 3	
Year	No.	No.	% of Year	No.	% of Year	No.	% of Year
R	45	1	2%	3	7%	4	9%
1 and 2	70	4	6%	4	6%	8	11%
3 and 4	65	17	26%	10	15%	27	42%
5 and 6	50	11	22%	18	36%	29	58%
ALL TASKS		Primary Markers - Tasks 1-3		Sec'dary Markers - Tasks 1-3		All Markers - Tasks 1-3	
Year	No.	No.	% of Year	No.	% of Year	No.	% of Year
R	45	7	16%	3	7%	9	20%
1 and 2	70	5	7%	11	16%	15	21%
3 and 4	65	20	31%	24	37%	33	51%
5 and 6	50	19	38%	25	50%	37	74%

Table 6.2: MARKERS BY ABILITY LEVEL ACROSS 7 YEARS IN 4 BANDS																					
TASK 1				Primary Marker - Task 1						Secondary Marker - Task 1						All Markers - Task 1					
Year	LA	MA	UA	LA	%LA	MA	%MA	UA	%UA	LA	%LA	MA	%MA	UA	%UA	LA	%LA	MA	%MA	UA	%UA
R	7	20	18	0	0%	1	5%	3	17%	0	0%	0	0%	1	6%	0	0%	1	5%	4	22%
1-2	15	33	22	1	7%	0	0%	0	0%	2	13%	2	6%	0	0%	3	20%	2	6%	0	0%
3-4	18	29	18	0	0%	0	0%	1	6%	0	0%	3	10%	4	22%	0	0%	3	10%	5	28%
5-6	16	16	18	0	0%	1	6%	2	11%	2	12%	3	19%	5	28%	2	12%	4	25%	7	39%
	56	98	76	1	2%	2	2%	6	8%	4	7%	8	8%	10	13%	5	9%	10	10%	16	21%
TASK 2				Primary Marker - Task 2						Secondary Marker - Task 2						All Markers - Task 2					
Year	LA	MA	UA	LA	%LA	MA	%MA	UA	%UA	LA	%LA	MA	%MA	UA	%UA	LA	%LA	MA	%MA	UA	%UA
R	7	20	18	0	0%	2	10%	2	11%	0	0%	0	0%	1	6%	0	0%	2	10%	3	17%
1-2	15	33	22	0	0%	1	3%	0	0%	1	7%	1	3%	2	9%	1	7%	2	6%	2	9%
3-4	18	29	18	0	0%	4	14%	2	11%	2	11%	7	24%	3	17%	2	11%	11	38%	5	28%
5-6	16	16	18	2	12%	7	44%	5	28%	4	25%	1	6%	5	28%	6	37%	8	50%	10	56%
	56	98	76	2	4%	14	14%	9	12%	7	12%	9	9%	11	14%	9	16%	23	23%	20	26%
TASK 3				Primary Marker - Task 3						Secondary Marker - Task 3						All Markers - Task 3					
Year	LA	MA	UA	LA	%LA	MA	%MA	UA	%UA	LA	%LA	MA	%MA	UA	%UA	LA	%LA	MA	%MA	UA	%UA
R	7	20	18	0	0%	1	5%	0	0%	0	0%	0	0%	3	17%	0	0%	1	5%	3	17%
1-2	15	33	22	0	0%	2	6%	2	9%	1	7%	1	3%	2	9%	1	7%	3	9%	4	18%
3-4	18	29	18	3	17%	11	38%	3	17%	1	6%	4	14%	5	28%	4	22%	15	52%	8	44%
5-6	16	16	18	0	0%	5	31%	6	33%	5	31%	6	37%	7	39%	5	31%	12	75%	13	72%
	56	98	76	3	5%	19	19%	11	14%	7	12%	11	11%	17	22%	11	20%	31	32%	28	37%
ALL TASKS				Primary Marker - Tasks 1-3						Secondary Marker - Tasks 1-3						All Markers - Tasks 1-3					
Year	LA	MA	UA	LA	%LA	MA	%MA	UA	%UA	LA	%LA	MA	%MA	UA	%UA	LA	%LA	MA	%MA	UA	%UA
R	7	20	18	0	0%	2	10%	5	28%	0	0%	0	0%	3	17%	0	0%	2	10%	7	39%
1-2	15	33	22	1	7%	2	6%	2	9%	4	27%	3	9%	4	18%	5	33%	5	15%	6	23%
3-4	18	29	18	3	17%	13	45%	4	22%	2	11%	13	45%	9	50%	5	28%	17	59%	11	61%
5-6	16	16	18	2	12%	8	50%	9	50%	8	50%	7	44%	10	56%	9	56%	12	75%	16	89%
	56	98	76	6	11%	25	26%	20	26%	14	25%	23	23%	26	34%	19	34%	36	37%	39	51%

LA : lower ability band MA : middle ability band UA : Upper ability band

Year R:	the first full year in the school, the oldest child being at most 5 years 9 months at the time of the interviews.
Years 1 and 2:	'Infants', aged between 5 years 9 months and 7 years 9 months.
Years 3 and 4:	'Lower Juniors', aged between 7 years 9 months and 9 years 9 months.
Years 5 and 6:	'Upper Juniors', aged between 9 years 9 months and 11 years 9 months.

In interpretive discussion of the data, these four bands, these phases of primary schooling, will be located against a background of institutional expectations and indices of arithmetical success within those phases. One felicitous consequence of the bandings is to achieve statistically-viable group sizes, and a degree of numerical parity between them (in fact, the sub-population sizes are 45, 70, 65, 50). The numbers of children giving a Marked response to each question are presented in the graphs (Figures 6.1 to 6.4) as percentages of the number in each band, so that comparisons between the bands may be made.

One feature of the graphs should be noted: that for each of the Tasks 1 to 3 (Figures 6.1 to 6.3), primary (spontaneous) and secondary (provoked) Markers are mutually exclusive. Therefore, the respective (dark and pale) columns may be validly stacked, the sum being the total number who Mark their response to that question. Figure 6.4, however, gives the size of the union of the sets (for primary and secondary Markers) in the first three graphs, and so the primary and secondary categories (unions) are no longer exclusive - it would be possible, for example, for a particular child to be recorded as primary on question 1 and secondary on question 3. Therefore the columns have not been stacked, but are displayed as three non-additive bars.

With one exception, the intention is to make comparisons (in the next section) across bands for each task (Figures 6.1 to 6.3) or for all tasks (Figure 6.4) rather than to compare task with task. The graphs have been scaled individually with this purpose in mind.

Occasional subsequent reference will be made to the additional data, tabulated in Figure 6.2 but not presented in the Figures 6.1 to 6.4. This was related to the attainment of the children in the domain of whole number concepts, categorised as below average/ average/ above average for their school year.

This assignment was entrusted to their class teachers, and the decision to delegate it to them was a deliberate one, in recognition of their deep knowledge of the children

Percentage of children in each age-band giving Marked responses to each of the three tasks

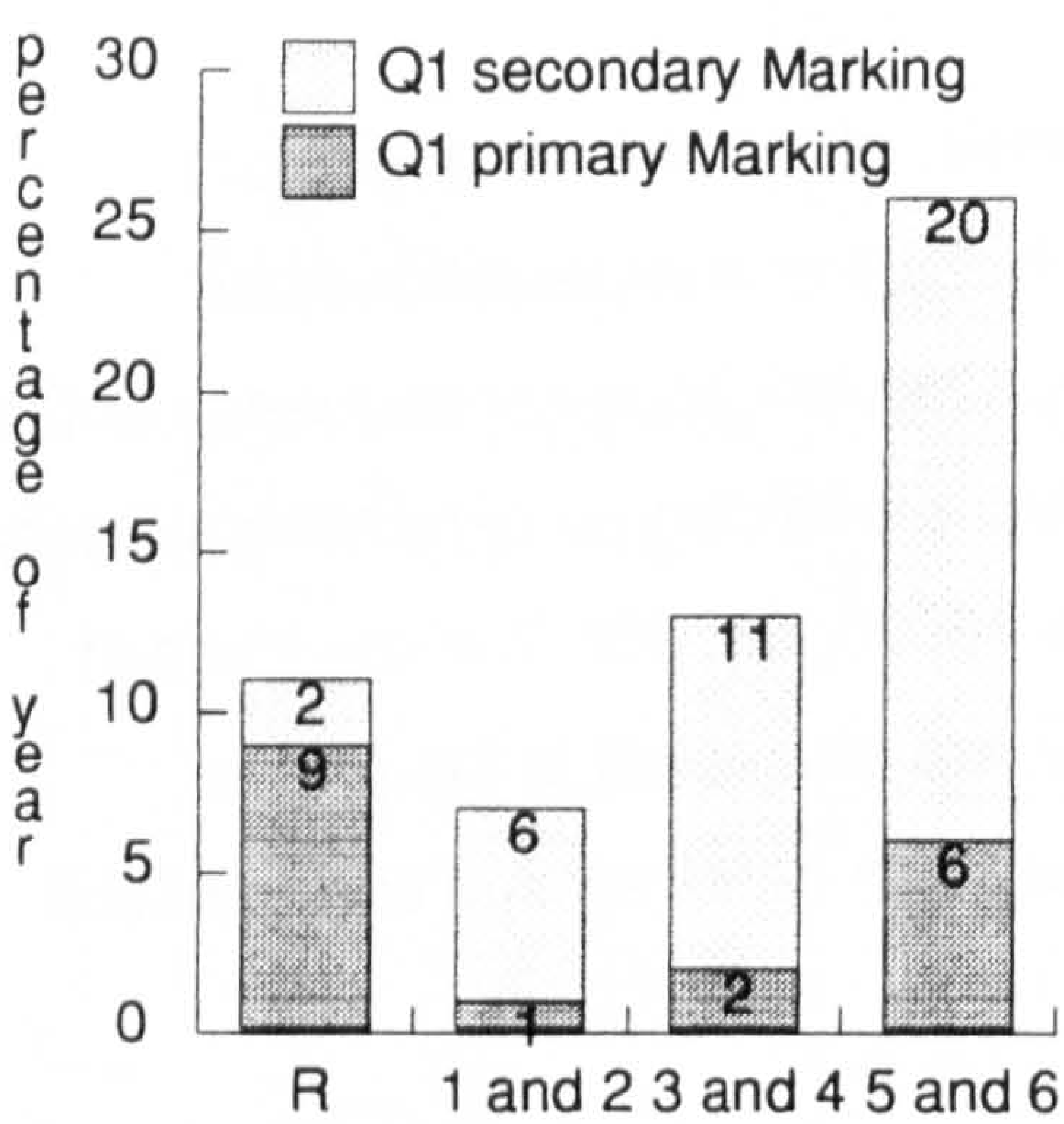


Fig. 6.1: Marked responses to Task 1

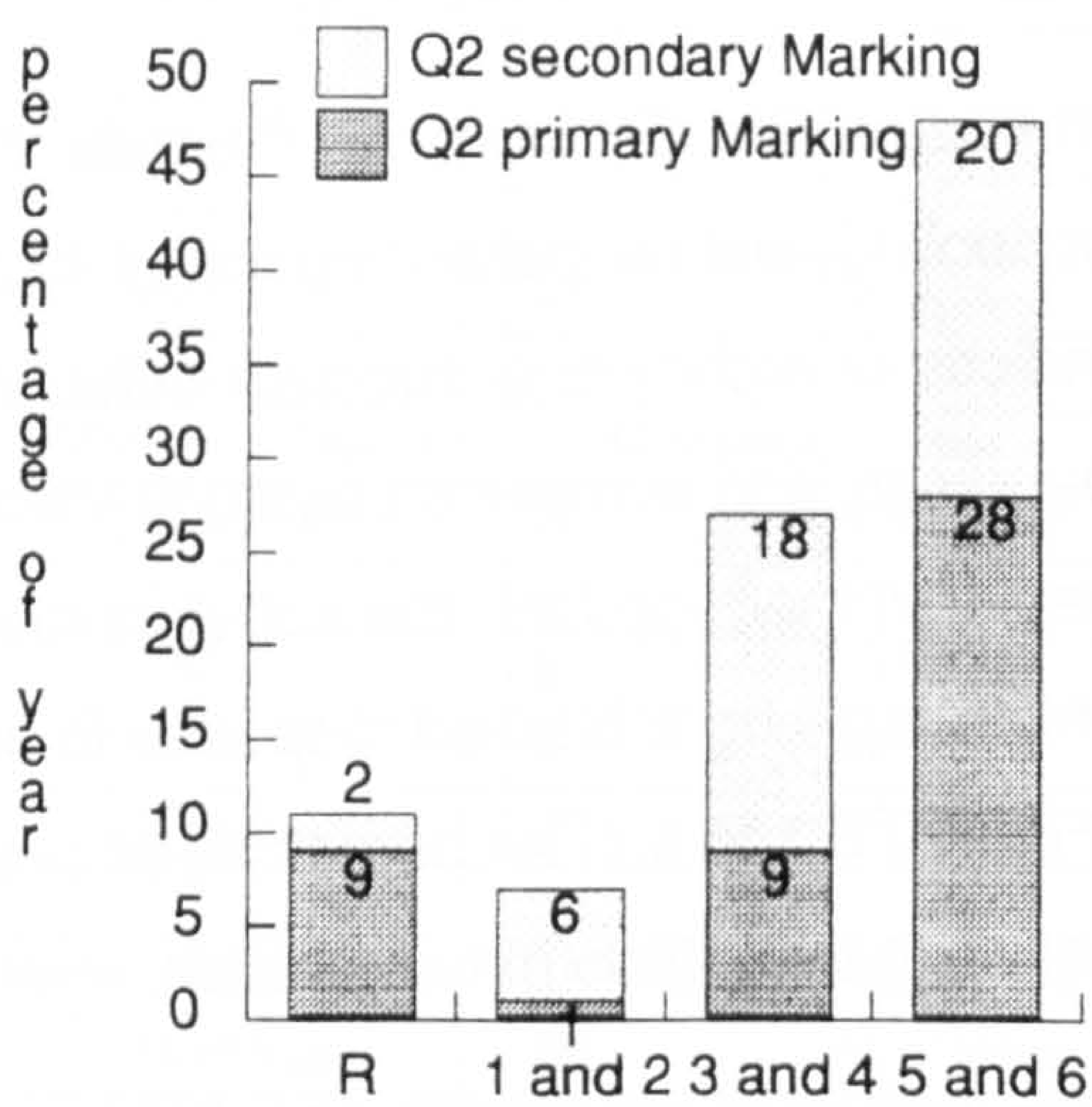


Fig. 6.2: Marked responses to Task 2

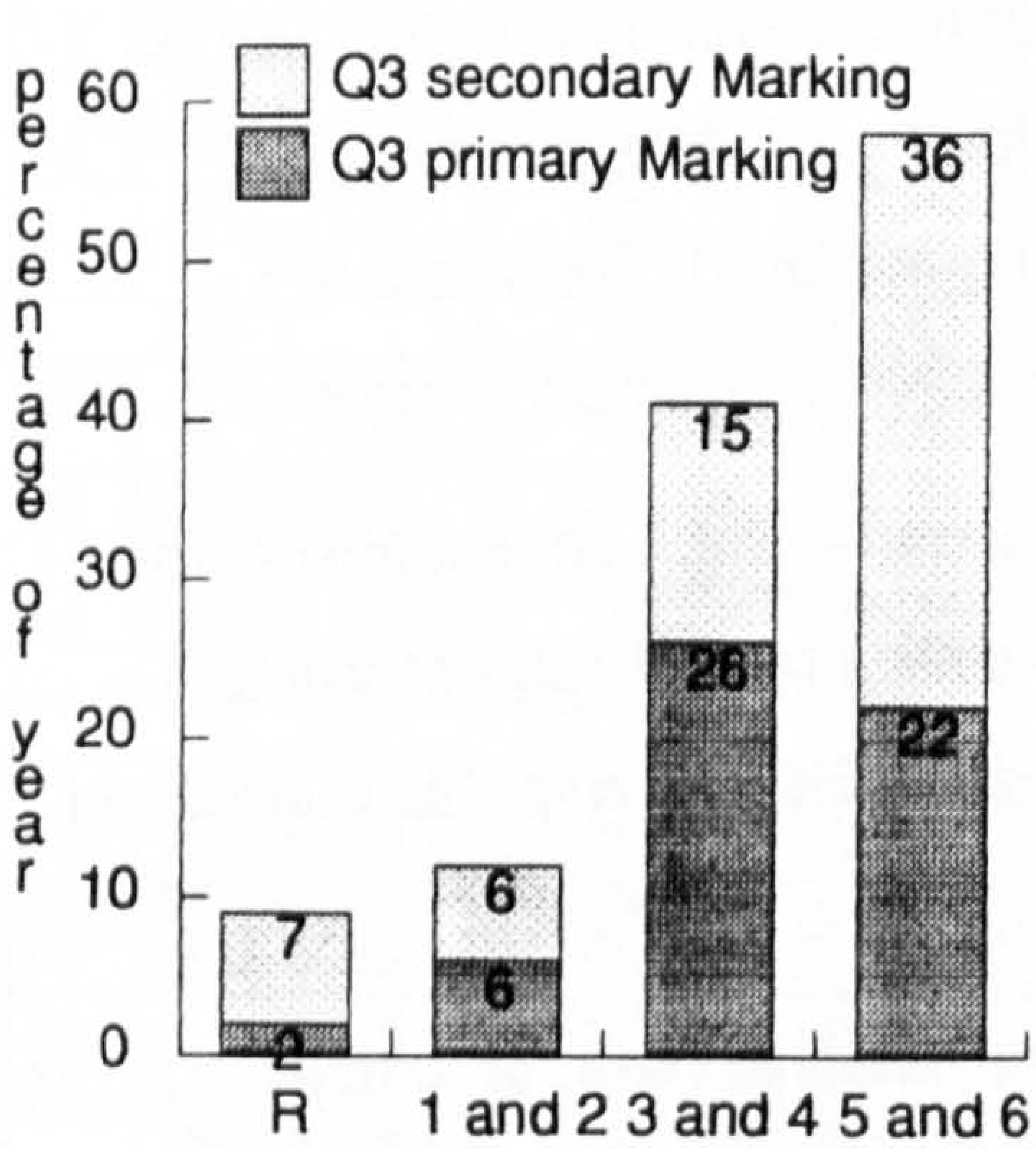


Fig. 6.3: Marked responses to Task 3

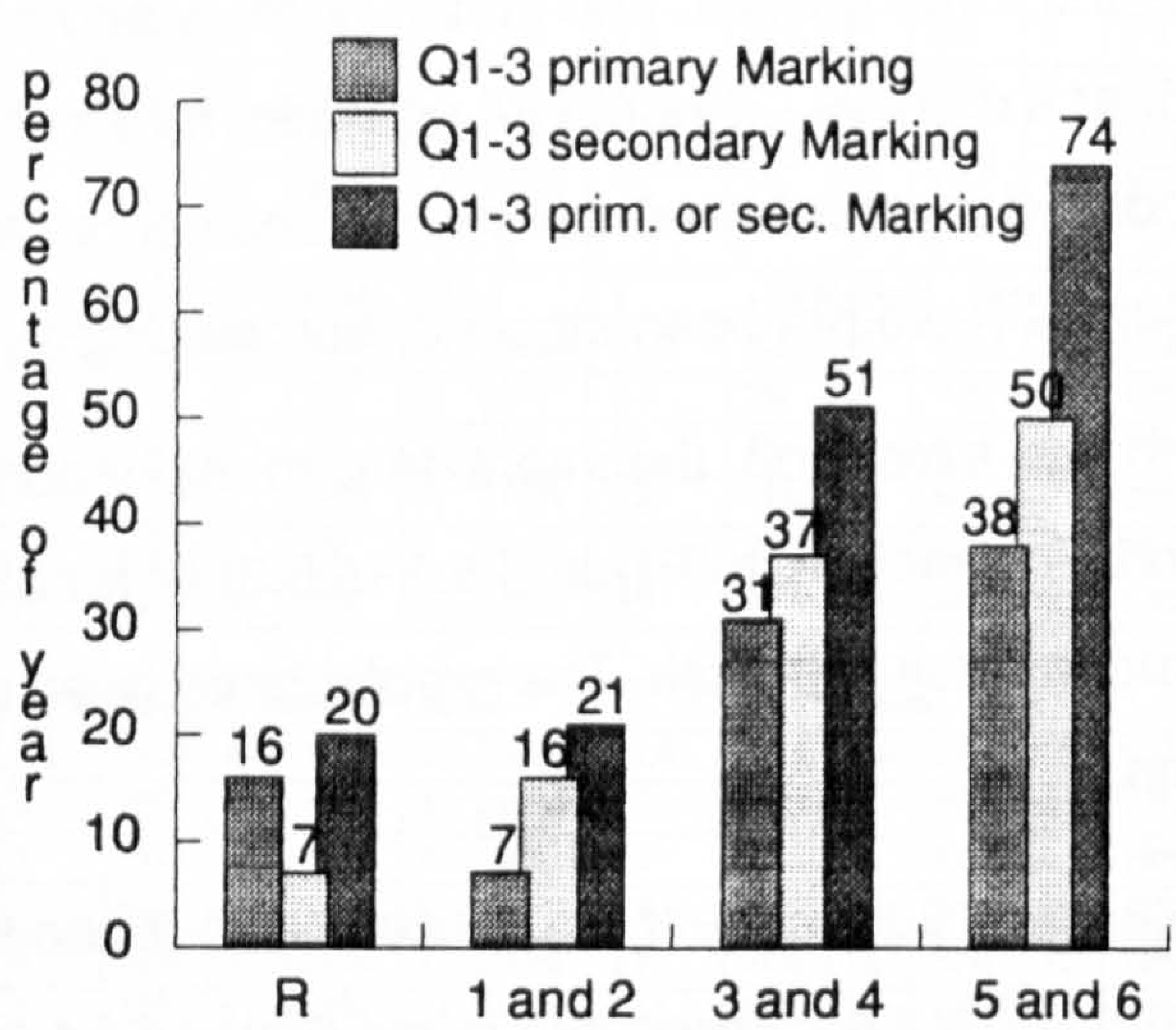


Fig. 6.4: Marked responses to Tasks 1, 2 or 3

whom they had taught daily for a whole school year. I regard this as more subjective but more reliable than a spuriously objective measure of attainment obtained, for example, from a Standard Attainment Task level. I gave no guidance about the proportion of children to be placed in each category. The relatively greater willingness of the teachers of the oldest (10- and 11-year-old) children to use the extreme categories presumably reflects their perception of their pupils' attainment as realised (or not) rather than potential. For understandable converse reasons perhaps, the Year R teachers proved reluctant to place their four- and five-year-olds in the lower-attaining category.

Table 6.2 re-presents the Marker data contained in Table 6.1, broken down into the three ability groupings for each age-band, labelled lower (LA), middle (MA) and upper (UA). Samples at the ability extremes were relatively small in each age-band, but any notable differences in Marked language between ability bands will be noted in the commentary which follows.

OBSERVATIONS

My observations here are restricted to some trends and features evident in the graphs. In particular:

1 The cumulative (stacked) Marked responses on Tasks 1 and 2 show a drop from the first band (Y R) to the second (Y1-2) with consistent increases thereafter.

2 This cumulative decrease over the first two bands (Tasks 1 and 2) is the result of very clear decreases in primary Marking between those bands.

3 With regard to secondary Marking only (Tasks 1 and 2), there is a consistent rise from band to band over the whole age range.

4 Likewise, on Task 3, the trend (minor inconsistencies apart) is of consistent increase with age.

5 Figure 6.4 indicates a greater tendency towards secondary Markers rather than primary ones in the last three bands (Y1 to Y6), but for the reverse in the youngest (Y R).

6 Taking Figures 6.1 to 6.3 together (but noting that the vertical scales are different): for the children in the last three bands (Y1 to Y6) there is an increasing tendency to Mark the response to each task in turn i.e. more Mark their response to Task 3 than that to Task 2, and responses to Task 2 are more Marked than those to Task 1. Again, this trend is not evident in the youngest group (YR).

A Note on Secondary Marking

I have already observed that, for any one of the three tasks, primary and secondary Marking are mutually exclusive, and that this justifies the accumulation of corresponding frequencies in the data and 'stacking' of the corresponding columns in the bar charts. In reading the data, however, it should be borne in mind that secondary Marking is *conditional* on absence of primary marking. Consider for example the results on Task 2: in Years 3 and 4, the 18% secondary Markers are drawn from the 91% non-primaries, and so represent **20%** of those who *could have* given secondary Markers. Whereas in Years 4 and 5, the 20% secondary Markers are drawn from the 72% non-primaries, and so represent **28%** of those who *could have* given secondary Markers. In this sense, the increase (adjusted to 40%) in secondary Marking between the third and fourth bands is even greater than the 'raw' percentages suggest (10% unadjusted). The same conclusion applies to the increase in secondary Markers from the second band to the third on Task 2, and indeed to Task 1 across the last three bands. This 'conditional adjustment' has the *opposite* effect between the first (YR) and second (Y1, 2) age-bands with regard to Tasks 1 and 2, because there is a sharp decrease in primary Marking between these bands, leaving a larger sub-population available for secondary Marking. For both Tasks, however, the increase in secondary marking is equally dramatic, and the inference of a consistent upward trend (Conclusion 3) remains valid.

INTERPRETIVE FRAMEWORK

It is reasonable to suggest that, in a broad sense, the data obtained from the tasks and interviews support the expectation (stated as an *a priori* conjecture earlier in this chapter) that ability to use linguistic Markers, and the tendency to do so, develops with age - at least over the years of primary schooling. It would not be a gross over-simplification, then, to say that the upward trends in the graphs point to the conclusion that children learn (or acquire increasing facility) to use vague language in order to convey attitudes and points of view, in particular uncertainty, in the primary years.

I shall proceed to propose a socio-linguistic developmental interpretative framework which would account for the upward trend, and which might also accommodate the unanticipated initial drops (from band R to band 1-2) noted above in Observations 1 and 2.

I suggest that the modal and hedging linguistic behaviours of the children in each band are related to three fundamental developmental dimensions.

1 The child's developing 'apprehension' of school - of the roles of the players (particularly teachers and their pupils) in the school situation, and the way that they relate to each other in learning situations. By 'apprehension' I mean the totality of factors such as: her (or his) perception of her role; how s/he construes (make sense of) a social environment which may have much in common with the home (e.g. being protected and cared for), but which evidently differs from the home in many respects (e.g. the number of individuals in a confined space, the relentless succession of tasks offered). Learning these and other essential differences between home and school is part of what Berger and Luckmann (1967) call 'secondary socialisation'. Primary socialisation takes place in the home, before school, when the family *is* the world, and reality is circumscribed by the child's experiences within the family. This reality is internalised and contributes to the child's sense of self in relation to a very small number of intimately-close significant others.

Secondary socialisation involves "the internalisation of institution-based sub-worlds [...] the acquisition of role-specific knowledge" (*ibid.*, p. 158). On going to school [Note 6.5] the child must learn how to become a pupil, one of many in a class, as distinct from "a certain mother's child". The internalisation which is part of secondary socialisation is weaker than that which takes place in primary socialisation; the child cultivates (some better than others) the art of 'role-distance', such that s/he is not wholly caught up in the identity which the institution imposes on them, but is able to distance themselves from it (and indeed, to question it) and perceive their role and that of others is a detached, formalised way. In particular, the child:

> apprehends his school teacher as an institutional functionary in a way he never did his parents, and he understands the teacher's role as representing institutionally specific meanings [...] Hence the social interaction between teachers and learners can be formalised. The teachers need not be significant others in any sense of the word. They are institutional functionaries with the formal assignment of transmitting knowledge [...] the consequence is to bestow on the contents of what is learned in secondary socialisation much less subjective inevitability than the contents of primary socialisation process. (*ibid.*, p. 161) [Note 6.6]

These factors contribute to the child's view of how s/he (and others) are 'positioned' within the institutional 'practices' (Walkerdine, 1988) which characterise schooling. In comparison with the integrated and deeply-embedded primary world of the home, in which the child is totally immersed, the world of school is more distant, more utilitarian, *more manipulable* (Woods, 1980, p. 13).

2 The child's developing (confidence in his or her) ability to produce desired behaviours within a variety of school practices. Such behaviours include an appropriate response to one's name at registration; reading aloud a limited passage of a book to an adult, with accuracy and minimum delay; giving 'correct' answers to questions (particularly arithmetic ones) to which the enquirer already knows the answer (Ainley, 1988, pp. 93-4; Walkerdine, 1988, pp. 54-5 and pp. 89-92). [Note 6.7] In particular, in the narrow context of the tasks presented to the children in this study, the child progresses over the first few years of schooling from a position where counting is a significant challenge, as far as accuracy is concerned, to one where it is a routine if necessary chore. I shall return to this matter later, with appropriate evidence.

3 The child's developing awareness of modal concepts and command of modal language. Hypothetical reasoning is, in the classical Piagetian formulation of cognitive development, a distinguishing hallmark of formal operational thinking. Such an account would essentially rule out modal concepts for most children in the primary years.

In their account of the growth of possibility notions, Inhelder and Piaget conclude:

> Compared to pre-operational or intuitive thought, concrete operational thought is characterised by an extension of the actual in the direction of the potential. [...] [However] the role of possibility is reduced to a simple potential prolongation of the actions or operations applied to the given context [...] They do not consist of imagining what the real situation would be if this or that hypothetical condition were fulfilled, as they do in the case of the adolescent. (Inhelder and Piaget, 1958, pp. 248-51)

Piérrault-Le Bonniec (1980) states the Piagetian position in a way which conveniently relates to the second and third three tasks in the study, suggesting (p. 76) that only at seven or eight years do children begin to have some idea of undecidability, and that the ability to reason from hypotheses is not acquired until age eleven or twelve. This would be likely to depend, however, on the extent to which the child's thinking was embedded in some real or familiar context (Donaldson, 1978, p. 76)

Moreover, Piérrault-Le Bonniec (*op. cit.*) has identified the presence and development of 'pragmatic modality' in young (age 3½ to 6) children - evidenced by the ability to assess what could, or could not, be made with a given set of materials. The combined picture, then, is of a development of modality from actions to perceived realities, and thence to imagined worlds. This corresponds to and is consistent with the finding that modalised utterances in early English child language predominantly express deontic meanings (actions related to obligation and permission); for example, 'can', 'could' and 'may' are used for action-oriented possibility, 'can' being significantly more prevalent than 'may' (Wells, 1979; Stephany, 1986, p. 390). The first epistemically modalised statements in pre-school children tend to occur about six months later than deontic meanings (Stephany, p. 396), but are still extremely rare in comparison.

Some insight on the development of Marked language in the early years of schooling is possible by sifting the results of a large-sample study of infant (age five to seven) vocabulary (Edwards and Gibbon, 1973) for the following data.

At age 5+, 'think' is present in the vocabulary, with lowish frequency index (FI) 0.3. (See Note 4.6 for the definition of FI). 'May' and 'might' first appear (in the Edwards and Gibbons sample) at 6+, with FI 0.46 and 1.53 respectively, with 'think' at 0.64. 'Perhaps' does not appear until age 7+, with low FI 0.18; by this age the FIs of 'may' and 'might' have roughly doubled in comparison with those at 6+, and that of 'think' has risen to 3.7. 'Maybe' does not appear in the data. [Note 6.8]

This is broadly consistent with data from a corpus of 250,000 English words spoken by American adults (Howes, 1966), which contains 66 occurrences of 'may', 102 of 'might', 139 of 'maybe' and 1034 of 'think'.

Whilst Approximators 'about' and 'around' are present in the infant corpus, the Edwards and Gibbon word-count must surely include non-hedging uses (such as "I'll tell you about my cat", and "Is mummy around?") and the FI values have little relevance here. Nor, for that matter, is 'think' only used epistemically, as a commitment-marker; adults, certainly, may use it in root form to make direct reference to cognition (as in the imperative "Think before you speak".)

Against this three-dimensional background I propose the following developmental narrative; the objective is to account for the trends in the data by reference to the interpretive framework that I have sketched.

INTERPRETATION OF THE DATA

The account of changes in Marked language and performance in the primary years (age four to eleven) is presented below in terms of three developmental phases, which I have termed Initiation; Suspicion; Approximation and Protection. These phases will be briefly characterised, then illustrated by extracts from the 230 task-based interviews.

Year R - Initiation.

In this first phase, the child's apprehension of school is relatively naive, and adult behaviour - questioning in particular - is taken at face value, without suspicion. Counting is a relative novelty, and the child is aware that her/his performance is sometimes faulty; this is hardly surprising, given the complexity of the process, as analysed by Gelman and Gallistel, Steffe *et al.* and Fuson. Whilst the task of enumerating 19 items may be *accessible* to a Year R child, a teacher will alert the child even when the count is substantially competent yet not entirely accurate; because accurate counting is a major goal, a targeted skill, in this phase of schooling. Moreover, Gelman (1977) suggests that only about one five-year-old in six can *accurately* enumerate a set of 19 items, given one minute to do so. In this sample of Year R children, it was actually one in five. The child (Year R) may well wish, therefore, to acknowledge to the interviewer, (as a primary Marker) some doubt about the answer s/he gives. In almost every case this is achieved with the plausibility Shield 'I think', with just a few epistemic modals, 'may', 'might' and 'maybe'. The child's Marking can be understood as straightforward cooperation, in effect observing the maxim of Quality is met, or serving notice that it may not have been met (Brown and Levinson, 1987, p. 164).

Tasks 2 and 3 are invariably approached by attempting a count of the sweets that are visible in the picture/the tube. Proportional reasoning (Task 3) and attempts to compensate for the possibility of hidden sweets (Task 2) are rare. The Piagetian caricature would suggest that the request by the interviewer for the number of sweets was accepted as achievable rather than a 'trick' question about something indeterminate or hypothetical.

It was the case that two-fifths of the Year R children who had (in advance) been informally assessed by their teachers as "above average for the year" with regard to whole number concepts, were recorded as giving a Marked response, whereas only two "average" and no "below average" Year R children used any Markers. This is not easy to interpret; it could mean that the lower attainers prefer not to draw attention to their uncertainty, or lack the linguistic means to communicate it. Perhaps they want the confidence to discuss their answer with the interviewer, or even lack awareness of its unreliability.

In each of the following transcripts, 'I' is the interviewer, 'C' the child.

Example 1: Boy aged 5:9 [M39]

The young boy in this transcript uses Marked language fluently.

M39:1 I: Can you tell me how many sweets there are on the plate?

[C counts aloud, points to each sweet in turn, takes 17 sec to reply]

2 C: One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen, twenty, twenty-one, twenty-two, .. twenty-three, .. twenty-four.

3 I: Twenty-four? Do you think there are exactly twenty-four?

4 C: **Maybe** not.

5 I: Why do you say that?

6 C: Because I **might** have counted two double.

7 I: [indicating photo] Right, now, there are some sweets in the glass. Can you tell me how many sweets there are in the glass?

[C counts aloud, touches the sweets in the photo, takes 6 seconds to reply]

8 C: One, two, three, four, five, six, seven, eight, nine.

9 I: And do you think there's exactly nine?

10 C: **I don't know, maybe** there's one more, because **I don't know** if I counted that one.

11 I: OK. Now, before we went into assembly I put ten sweets in that tube, so I know there are ten sweets in there. Can you tell me how many sweets there are in that tube?

[C counts aloud, touches the outside of the tube, no comparison of the tubes, takes 22 sec to reply]

12 C: One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen, ... twenty, twenty-one, twenty-two, twenty-three, twenty-four, twenty-five.

13 I: Twenty-five. And do you think there are exactly twenty-five?

14 C: **Maybe** not because there's so much.

The "How many?" question is received by this boy as a straightforward invitation to count. He has command of the stable-order principle [M39:2], but applying the one-one principle to a disordered set of this size presents difficulties for him. He does not touch the sweets one by one, but points to them - a partial internalisation. His ability to partition the set (Gelman and Gallistel, 1978, p. 77) into counted and to-be-counted subsets is faulty *and he knows* it is sometimes faulty [6]. In fact, he has double-counted three of the sweets. His awareness of the same difficulty, with partitioning, prompts his Marked response [10] to Task 2. In this case he is suspicious of omission whereas before it was duplication. For Task 3, his order-stability in reciting the 'tags' 1 to 25 is once again faultless; he expresses his uncertainty in his answer with the same epistemic hedge [14] 'maybe' (not). The meaning behind his voluntary explanation "because there's so much" is unclear, but could suggest a tacit estimate of some sort which does not accord with his count. Alternatively, he may in effect be acknowledging that he regularly makes a partitioning error when the set to be counted is as large ("so much") as this one.

In each case, he applies the cardinality principle unerringly. In fact, his competence on the first and third tasks confirms Fuson's ordering of mastery of the principles for sets of this size. What is very clear from this transcript is that he takes a face-value view of the interviewer's three questions, that he counts rather than estimates, but that he is quite uninhibited about notifying the interviewer that his (the child's) answers may be unreliable. His attitude is something like: "That's what I make it but I know from experience that I may have made an error. That's simply the way things are when, like me, you're a novice at counting".

Years 1 and 2 - Suspicion.

The child's apprehension of school includes the sense of being scrutinised by curious adults, of the existence of 'testing' questions. S/he is now expected to enumerate small sets routinely, and has built on this for the purposes of addition and subtraction of whole numbers. The result is a manifest reluctance to use primary Markers in response to the first two tasks (perhaps because "he wants to know if I can get it right"), but the interviewer's probe ("Exactly?") may release an acknowledgement of uncertainty, using the same Plausibility Shields (predominantly 'I think' with a few 'maybe's) as the Year R child. The more subtle linguistic ability to use Approximators as epistemic hedges, for self-protection by vagueness, has not yet been developed.

Task 3 may seem bizarre, quite unlike routine practical or text book counting exercises; it gives rise to more hedged responses, though not in the low attainers, who are most likely to hedge on Task 1 - like the younger, higher-attaining Year R children.

Example 2: Boy aged 7:5 [reference number 55]

The transcript is chosen for absence of Marked language.

M55:1	I:	Can you tell me how many sweets there are on the plate?
		[C counts aloud, touches each sweet, takes 23 sec to reply]
2	C:	[quickly] One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, .. oh, [restarts counting, now slower, placing sweets on the table whilst counting] One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen.
3	I:	And do you think there are exactly nineteen?
4	C:	What?
5	I:	Do you think there are exactly nineteen?
6	C:	Er .. [pause, then recounts into hand] One, two, three, four, five ... One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen.
7	I:	So do you think there are exactly nineteen?
8	C:	Yes.
9	I:	Right ... can you tell me how many sweets there are in the glass?
		[C turns over photo to try and see the "back" of the glass; counts aloud, takes 20 sec to reply]
10	C:	Oh! One, two, three, four, five, six, seven, eight, nine, ten, eleven. ... one, two, three, four, five, six, seven, eight, nine, ten, eleven, eleven.
11	I:	And do you think there are exactly eleven?
12	C:	Yes.

His first recount [M55:2] is presumably a self-correction after he suspects that his partitioning has gone wrong. There is no problem with the order of the tags, and the count is in fact accurate. The interviewer's probe [3,6] is immediately taken to be a suggestion that he has mis-counted. Instead of hedging, he re-counts the set [6]. The indeterminacy of the number of sweets in the glass is either not perceived or not acknowledged [12]. Again, he counts the (visible) sweets unprompted.

This child's uncertainty, his sense that he may make mistakes when he counts, is not conveyed in Marked language, but by his inclination to re-count. His approach to Task 3 (below) is much the same, until he admits some (unspecified) difficulty. 'Can't' [14] is, of course, a root modal associated with capability. The interviewer's prompt [15] precedes proportional thinking and an un-Marked response.

13	I:	Now, I counted some sweet into here, and I know there are 10 sweets in there [...] Can you tell me how many sweets there are in that tube?
14	C:	One ... two ... three ... four ... can't do this! One, two, three, four, five, six, seven, eight ... one, two, three, four, five, six, seven, eight, nine, ten ... nine, ten ... eleven ... one, two ... I can't figure it out right.
15	I:	How many do you think there are? [pause] There are ten in there OK [...] We know that there are ten in there.
16	C:	I know that ... [*compares tubes without counting*] ... twenty.
17	I:	Why do you think that?
18	C:	Because it's half.
19	I:	OK. Excellent [...]

Years 3 to 6 - Approximation and Protection.

There is developmental continuity within this phase rather than qualitative change. The account which will follow characterises a child who has moved some way along that developmental continuum. In fact, the 'low ability' children in this phase used Markers consistently less (significantly less on Task 3) than their higher-attaining peers - in many respects their response was more like that of the younger, higher-attaining Year 1-2 children.

The child has a well-developed apprehension of her role in the practice of education. She realises that teachers' - including researchers' - questions about mathematics are usually not simple requests for information (Walkerdine, 1988, p. 61). Indeed, she may have become quite expert at eliciting from the teacher the very information that she (the child) was originally asked to produce, or an easier question with the same answer (MacLure and French, 1980). This kind of pupil behaviour is a parody of 'thinking', a strategy with a long pedigree:

> Each time I had to think of a question easier and more pointed than the last, until I found one so easy that she would feel safe in answering it [...] In fact, she was not even thinking about it. She was coolly appraising me, weighing my patience, waiting for that next, sure-to-be-easier question. I thought, 'I've been had!' The girl had learned how to make all her previous teachers do the same thing. (Holt, 1969, p. 38)

On the other hand, the child is confident in her ability to count; of the 108 Year 3 to 6 children who did choose to count the 19 sweets in Task 1, about two-thirds did so accurately, and four-fifths of the remainder obtained 18 or 20. If such a child does in fact count the sweets in Task 1, then a primary Marker is unlikely. On the other hand, she may judge that the interviewer is not interested in the precise number of sweets on the plate, and offer a primary Marked estimate (there is a corresponding rise in the fourth band), or a response which is tagged (by intonation) with a question mark.

She recognises that some quantities are indeterminate; on Tasks 2 and 3 s/he will realise, despite the awareness of 'testing', that the interviewer cannot sensibly expect her/him to give precise answers to these "How many?" questions. If his/her first answer is un-Marked - as if s/he were guessing how many sweets in a jar at a fête - she will readily admit to uncertainty when asked if that answer is exact. S/he may recognise the proportional reasoning Task 3 for what it is, and (correctly) have some confidence that there are exactly 20 sweets in the second tube. This is strongly reminiscent of many eleven-year-olds involved in Assessment of Performance Unit practical testing, who were very resistant to the notion that a 20 gram weight was being counterbalanced by 21 (rather than 20) plastic tiles (Joffe, 1985, p. 21A). These pupils tended to 'demand a recount' in order to obtain the preferred 'round' answer. At the same time, however, Clayton (1992) observes that pupils will frequently actually avoid round numbers when asked to estimate numerosity. He calls this tendency the 'jelly-baby effect':

> Guess the number of jelly-babies in the jar and whoever is closest, wins the prize. Some pupils appear to believe that the person in charge would not have a 'round' number of jelly-babies so they guess a number close to but not exactly the round number. (p. 117)

At the same time, s/he has developed competence to deploy Approximators such as 'about' (as well as modal auxiliaries) as epistemic markers, to introduce vagueness for protective purposes.

Example 3: Boy aged 8:1 [reference number 165]

M165:1	I:	Can you tell me how many sweets there are on the plate?
		[C points to each sweet, counts silently, takes 19 sec to reply]
2	C:	Nineteen.
3	I:	Do you think there are exactly nineteen?
4	C:	Yes.
5	I:	And how do you know that?
6	C:	I counted up in twos.
11	I:	OK. Can you tell me how many sweets there are in the glass?
		[C points to sweets, counts silently, takes 20 sec to reply]
12	C:	Ten.
13	I:	Ten. And do you think there are exactly ten?
14	C:	No
15	I:	No? Why not?
16	C;	Mm ... 'Coz there **might** be some more at the other side.
17	I;	I've put ten sweets in this tube here OK ... can you tell me how many sweets there are in that tube?
		[C touches (turns tube), counts silently, compares tubes, takes 31 sec to reply]
18	C;	**About** twenty-one.

Quite confident of his internalised count of the sweets, he has no cause to Mark his responses [2, 4, 6] to Task 1. On the other hand, the indeterminacy of Task 2 gives rise to the secondary epistemic modal 'might' [16]. His approach to task 3 is cautious and unhurried. The interviewer judged that some comparison was made between the tubes and hence that there may have been some proportional inference. The eventual answer [18] includes an Approximator 'about'. It is not possible to judge from the transcript whether this hedge is epistemic or root. The actual estimate of 21 (which is not a 'round' number) appears to be an instance of the 'jelly-baby effect' influencing the choice of estimate.

Example 4: Girl aged 10:6 [reference number 222]

1	I:	Can you tell me how many sweets there are on the plate?
		[C doesn't count, takes 2 sec to reply]
2	C:	**About** twenty?
3	I:	Now, can you tell me how many sweets there are in the glass?
		[C doesn't count, takes 1 sec to reply]
4	C:	Ten.
5	I:	And do you think there are exactly ten?
6	C:	No! [laughs] **not exactly**.
7	I:	Why did you say that?
8	C:	'Coz it's **not actually** ... it doesn't look like ten ... well I just guessed.
9	I:	OK. Now, I've put ten sweets in this tube ... can you tell me how many sweets there are in this tube?
		[stares, compares tubes, takes 1 sec to reply]
10	C:	Twenty.
11	I:	And do you think there are exactly twenty?
12	C:	**About** twenty-five ... **or** twenty.
13	I:	And what makes you say that?
14	C:	'Coz it **looks like** half, twice as much as in there.

This pupil approaches each task with mathematical confidence and social maturity. On the video she looks relaxed, at ease with the interviewer. She knows exactly when an estimate will suffice and, indeed, when nothing else is possible. For Task 1, she instantly judges that an estimate will meet the requirement of the maxim of Quantity. Her Approximator qualifies a suitably round number [2] with the force of a root hedge (as if to say "This is as much as you need to know"). She just laughs at the suggestion that her vague use of ten [4] as a 'cognitive reference point' (Rosch, 1975) should be taken to be anything other than an approximation. For Task 3, she is explicit that proportional reasoning is the basis of her rounded estimate.

Her response [12] to the interviewer's probe [11] needs closer examination. Channell's analysis of the approximator form *n or m* (1994, pp. 53-58), drawing on Crystal's book on intonation (1969), reveals a subtle prosodic distinction between a binary alternative and a vague range of alternatives. Consider the difference between the two following questions (scenario: parent to child about to go on a school trip):

I:6.1 Will you need five or ten pounds?

[meaning: which of two bank notes shall I give you?]

I:6.2 Will you need five or ten pounds?

[meaning: is that the kind of amount you'll need?]

The semantic distinction is achieved by prosodic marking of the questions. In the first case, the two alternatives (*five, ten*) are both stressed and nuclear (i.e. have maximal prominence). In the second, *five or ten* is a single, unstressed tone unit and *pounds* is nuclear. Channell also points out that, for the approximative use of *n or m*, n must always be less than m. This is not the case in [M222:12]. Sure enough, replaying the videotape of the interview confirms that the alternatives *twenty-five, twenty*, are not parts of a single tone unit in the utterance. They are quite separate, the first being stressed slightly more than the second. The impression conveyed is that she considers revising her estimate (to 25, the next 'round' number in this range) but changes her mind perhaps on grounds of comparison and proportional reasoning.

GENDER DIFFERENCES

An expectation that girls/women hedge more than boys/men is commonplace (Lakoff, 1975). This expectation is related to a supposition that males are more assertive and confident than females in a number of social situations. Clayton (1992) identifies estimation as a risk-taking school activity, and concludes that boys, on the whole, approach it with greater confidence than girls, markedly so in the years of secondary schooling. Insofar as modals and hedges can achieve a measure of protection for the speaker against accusation of error, one might expect some tendency for the girls in the whole-school study to use these Markers more than boys in their responses to the three tasks.

The data from this experiment do not, in general, bear out that expectation. It turns out that one-fifth of infant (4-7) boys used some Marked language, as did one-fifth of infant girls (but see below). For junior (7-11) children, the proportion was three-fifths for both boys and girls, with 41% of boys Marking one or more responses (n=125) and 41% girls (n=105) over the whole school population.

Suspicious that such a global report on gender differences (or the lack of them) may overlook some fine distinction, I submitted the data to a number of tests for differences of a more detailed nature. Expecting to find differences, if there were any, among the junior pupils, I concentrated my efforts on the 115 pupils in Years 3 to 6. Once again, the proportions of boys and girls exhibiting the linguistic behaviour were much the same in every case. For example:

Giving at least one primary Marked response:	Boys 36%	Girls 31%
Giving at least one secondary Marked response:	Boys 42%	Girls 43%
Giving at least one Marked response to Tasks 2 or 3:	Boys 56%	Girls 57%

The only significant gender difference which I have uncovered from the data concerns the proportions of boys and girls in Year 1 and in Year 2 who give at least one Marked response:

At least one Marked response (Year 1):	Boys 11%	Girls 33%
At least one Marked response (Year 2):	Boys 30%	Girls 7%

The fact that the bias is reversed from Year 1 to Year 2 obscures the difference when the Infant data is considered as a whole. (The Year R data is more balanced with some bias towards Marked language in boys.) The clear bias in Year 1 and the reverse bias in Year 2 stands out as the only significant gender difference that I was able to detect, yet I am unable to account for it. Given the improbably sudden switch of bias, and given the numbers of pupils involved in each year (Y1: n=39; Y2: n=31), I am inclined to advocate a replication of the study with children of this age before asserting any firm gender-specific conclusions. Clayton's findings would lead one to expect that gender differences in Marked language might become more evident in the years of secondary schooling (after age 11).

MODAL AUXILIARIES

The term 'Marked' has been introduced to include modal auxiliaries and hedges. Initially, hedges were to have been the sole focus of study, but the epistemic similarities between modals and hedges motivated the inclusion of modal auxiliaries. The two classes of language clearly overlap in adverbial forms such as 'possibly', 'maybe'. Only 19 children in the sample of 230 used modal language that was not also recorded as a hedge. These 19 were predominantly in the third age-band (Year 3 and 4) and in nearly every case the modal verb used was 'might'. The sole exception was a girl in Year R, aged 5½, judged to be mathematically 'above average', who made a fully internalised count of the sweets in response to each task (with no external manifestation of the two 'local correspondences'). Her eventual answer to each of the three "How many?" questions was itself marked as a question with rising intonation, but with no secondary hedge on the first two tasks. For the third she gave the answer 23, and in reply to the prompt "Do you think there are exactly twenty-three", she answered, "There may be one or two more", with prosodic marking indicating epistemic approximation.

Of the remaining 18, all of whom used 'might', the following are typical:

"I might have counted two double". (Year R boy, Task 1)

"There might be some more". (Year 1 girl, Task 2)

"There might be some more on the other side". (Year 3 boy, Task 2)

"There might be less". (Year 3 boy, Task 3)

"It might not be doubled". (Year 4 girl, Task 3)

It is interesting to note that, whereas occurrences of 'may' and 'might' were rare, in every case they arose as *secondary* Markers, in response to the interviewer's prompt: the examples above convey this reactive uncertainty well.

The adverbial hedge/modal 'maybe' is also used by 13 children (3 of whom also use 'might'), and is more evenly distributed over the age-range in the school.

PROSODY

A decision (arguably a conservative one) was made not to record as a hedge a third type of response, namely uncertainty expressed in assertions which are offered as questions, usually through rising intonation (such as "Twenty?"). I am acutely conscious that such responses are contenders for inclusion, as Shields. Stubbs (1986, p. 21) observes that tag questions "allow statements to be presented as obvious, dubious, or open to challenge". What is true in this respect of explicit tags (such as "..., isn't it?") must also be true of tacit, prosodic tag questions.

Of the 230 children in the sample, 52 gave numerical answers marked by rising intonation; of these 30 Marked one or more of their responses in some other way which was included in the analysis. The remaining 22 remain outside the modal/hedge consideration of this study. The oldest year-group (Year 6) accounts for none of the 22, who are fairly evenly distributed across the earlier Years R to 5. It might be interesting to include these in the 'Marked' category and re-analyse the data, but I would not expect this to significantly affect general trends given the conspicuous rise in Marked responses in the fourth band (Years 5 and 6) under the operational definition used in the study.

SUMMARY

In this chapter, I have attempted to interpret and to impose some rationale onto the data obtained from short, task-oriented interviews with the entire pupil population of one primary school. I began with the expectation that the data would support a hypothesis that the ability to use linguistic Markers, and the tendency to do so, increases consistently through the primary years. These Markers (hedges and modal auxiliaries) serve epistemic and root purposes, conveying either the speaker's uncertainty or their awareness that an estimate was appropriate (in fact, essential in the case of the last two tasks). The outcome of the study includes an account of some developmental aspects of conveying uncertainty in mathematics, including social aspects which would account for a dip in unprompted Marked language shortly after the child's initiation to schooling.

I have argued, mainly in Chapters 1, 2 and 5, that in the context of mathematical activity, uncertainty is a normal state, potentially a creative one. A creative objective for teachers could be to recognise it and work with it (as well as with their own epistemological and pedagogical uncertainties) rather than to seek - in the short term, at least - to deny or eliminate it. In conclusion, it is only proper, and it is certainly appropriate, to admit to some uncertainty about the account I have given, and to seek resonance (or correction) by offering it for inspection. Some words of Valerie Walkerdine convey my own position:

> "These thoughts are speculative, but I suggest that it is in this direction that our analysis should go if we are to understand more fully the responses given by children." (1988, p. 59)

Whilst it is clearly the case that the use of Marked language has, in some sense, to be learned, it is not so clear how that particular linguistic competence is acquired, a competence that includes a modal dimension which lies at the core of the communication of propositional attitude. The unexpected outcome of this particular study, in the context of estimation activity, is that children may be socialised into suppressing this aspect of their linguistic competence until they discover, or assert, that - in some mathematics classrooms at least - it is alright to be wrong.

CHAPTER 7: PRAGMATICS, TEACHING AND LEARNING

> Language which fulfils a genuine need ... will ultimately find an audience ... to whom it is not obscure. (J. F. Wallwork, 1969)

This is the last chapter in which I will introduce new empirical transcript data. Whereas the data in the earlier Chapters 4 to 6 are unified personally by informant (e.g. Susie) or thematically by phenomenon (e.g. hedges), those to be presented and considered here are more diversified in origin and nature. There is another important distinction: whereas the data in the earlier chapters were collected in the course of research interviews, the data presented here are transcripts of a variety of teaching and learning situations.

The plan of the chapter is as follows:

(I) to explain the formation and work of an 'Informal Research Group' of school teachers which I convened to consider and act on my earlier research findings on vagueness in mathematics talk, and to provide further data for my consideration;

(II) to apply the complete linguistic toolkit that has been assembled to the analysis of a number of 'Cases' - transcripts or fragments - demonstrating further the role of vagueness in mathematical discourse by reference to a variety of episodes in teaching and learning. These cases come from a number of sources, principally the Informal Research Group. Additional data, collected but unused in this thesis, are listed in Appendix 1.

THE INFORMAL RESEARCH GROUP

For about four years I pursued my thoughts and enquiries for this thesis in relative isolation; at least, in the isolation of the 'ivory tower' community of academic researchers, research seminars and conferences. My experience of sharing my ideas in that community had been sufficiently encouraging (and corrective) for me to gain in confidence about my methods and my findings, that I wanted next to share them with schoolteachers. My expectation was that teachers preoccupied with the administrative and managerial demands of daily classroom teaching would be looking for some relevance in my work to their day-to-day concerns.

On 23rd February 1994 I gave a research presentation (Midland Mathematics Education Seminars) at Warwick University. After the seminar, Leone Burton encouraged me to convene a group of schoolteachers and to share my findings with them. In the Autumn, I broadcast an invitation to primary and secondary schoolteachers (direct-mailed with the College INSET programme) to join an "Informal

Research Group" which I would convene early in 1995 to consider aspects of language and mathematics. About a dozen teachers indicated interest. I wrote to them:

> Many thanks for expressing interest in joining the small Informal Research Group which I plan to convene next term.
>
> As the flier indicated, the focus for the research is on linguistic pointers to uncertainty in the learning of mathematics. My methods to date have involved recording and transcribing "mathematical conversations" with children, individually or in pairs. Such conversations have focused on problem solving or open-ended tasks of various sorts, where uncertainty can be raised by the need for prediction or generalisation. Another area of uncertainty - more restricted and more fully researched - has to do with estimation. Words and phrases such as *around, about, maybe, I think* are indicative of uncertainty, and are used in different ways to serve a variety of purposes for the pupil. Some literature from linguistics can be applied to understand how this works i.e. has the desired effect on the hearer.
>
> Within the informal group I should like to share my findings to date, and in time to encourage group members to collect further examples of the same kind of language in mathematics classroom. Such examples might be generated in individual or small group discussion with you, or in whole-class discussion. One of the things we can do in the group is to propose and consider suitable tasks, having prediction and generalisation in mind. The important factor would be that your examples are collected in the context of "normal" classroom teaching.

The group was self-selecting. This is not to claim that it was in any sense representative of some 'generality' of schoolteachers, but nor was it was selected by me to be cooperative or compliant. In the event, it was certainly affirmative as to the value of the work that I was engaged in.

Three evening meetings of the group took place in January-February 1995. About eight teachers attended each meeting. At these gatherings I gave an account of my principal method, contingent interviewing, illustrated by transcripts of my mathematical conversations with children. I went on to explain how I believed that hedges were an important tool in the children's linguistic repertoire, principally as a Shield (epistemic hedge). I distributed some papers for further reading. These were Ginsburg (1981), on clinical interviewing; a draft of Rowland (1995), the forerunner of Chapter 5; and Rowland (1994a), which I later developed into Chapter 6. Finally, we discussed possibilities for tasks that would be an appropriate basis for mathematical conversations between them and their pupils. They then undertook to carry out, record and transcribe such conversations before our next meeting. Two members of the group opted out at this stage. Six came to a Saturday morning preliminary 'reporting back' session in March, and again to a further meeting on a Saturday morning in June, by which time their transcripts were ready. I hope that some jointly-authored publication will come out of the work of the group, and have encouraged members to write in preparation for it.

One aspect of this group, which delighted and surprised us all, was the age-range spread which it represented - from a reception class teacher to someone teaching A-level students in a sixth-form college. One member of the group was in her first year of teaching, two in their third, and the others had been teaching for some years. We rapidly discovered that the language of classroom mathematical discourse provided a focus for our discussion and close attention, a focus that facilitated and energised mutual effort and understanding. Indeed, the shared cross-phase dimension added greatly to the interest of the work of the group. Remarkably, no contribution seemed irrelevant to the professional concerns of all group members. For me, this was in pleasing contrast with much of the subject-content based INSET I have been 'contracted to deliver' over the last five years, where many teachers seem to judge relevance in terms of the text of the latest National Curriculum programme of study for mathematics at their own key stage.

Equipped with my eclectic set of approaches to discourse, I now proceed to examination of eight 'cases'. The teachers featured in the first five cases are members of the Informal Research Group. Some cases are based on quite lengthy transcripts, and my consideration is forced to be selective since each transcript could be the basis of an extended analytical essay. My purpose here is to give a demonstration of the linguistic methods and evidence of consistency with the observations from my own contingent interviews with children, as presented in the earlier chapters.

CASE 1: HAZEL

Hazel is a primary school teacher in her third year of teaching. In the transcript (IRG3) she is talking with two ten-year-old girls in her class, Faye and Donna. Hazel indicates that "both are able mathematicians who often work together".

The conversation is an exploration of the difference between b^2 and ac, where a, b, c are consecutive terms of an arithmetic sequence. It falls into four episodes:

Episode 1: Investigation of the case when the common difference is 1 [IRG3:1-61]

Episode 2: Investigation of the case when the common difference is 2 [IRG3:62-105]

Episode 3: Investigation of the case when the common difference is 3 [IRG3:106-120]

Episode 4: Search for a higher-level expansive generalisation (Harel and Tall, 1991: see Chapter 1) which includes the three generalisations arrived at inductively in the previous episodes as special cases [IRG3:121-160].

First, observe that in every case Hazel's instructions and requests to the two girls are presented as indirect speech acts, for example (there are many):

17	Hazel:	Shall we try it out and see what happens? Do you want to each choose your own set of consecutive numbers?
66	Hazel:	Right would you like to try out with ten, twelve and fourteen one of you and the other one can try another jump.
130	Hazel:	Can you tell me what the difference in the answers of the two sums that, the two multiplications you're doing would be when you have a difference of four between each number?

IRG3:17 and 66 are on-record FTAs, with redressive action ('orders' presented as questions) with regard for the children's negative face, as Hazel imposes on their personal autonomy of action. These are conventionally polite, indirect speech acts (like "Can you pass the salt, please?). She believes that the investigation will be a worthwhile, educative experience for them with a potentially stimulating outcome. Nonetheless she recognises the risk-taking which is inherent in her quasi-empirical approach, and that she requires their cooperation as active participants in the project as they generate confirming instances of generalisations-to-come. In [17] she says "Shall we try it out?", the plural form including and identifying herself as a partner in the enterprise. In [130] she probes for a prediction (related, possibly, to an as-yet unarticulated expansive generalisation) and realises the threat to the girls' positive face - what if they fail to make a correct prediction, will their reputation as "good mathematicians" be dented? [130] respects their positive face, and the indirect modal form redresses the on-record FTA. The form of her question - "Can you tell me ...?" is precisely that which I chose for the whole-school developmental study. [Note 6.4] Quasi-empirical teaching, inviting conjectures and the associated intellectual risks, is unimaginable if the teacher is not aware of the FTAs that are likely to be be woven into her/his questions and 'invitations' to active participation. Redressive action dulls the sharp edge of the interactive demands that this style places on the learner. For Hazel, notwithstanding her authority in her own classroom, the indirect speech act has become a pedagogic habit, which even extends to non-cognitive requests:

41	Hazel:	Would you like to go and get one [a calculator] Donna?

Children are likely to be less sensitive, to be more direct:

42 Faye: Get two.

The children make frequent use of Shields as epistemic hedges. Early in the conversation Faye [IRG3:9] observes a difference of 1 between 10x12 and 11^2. Somewhat precipitately, perhaps, Hazel asks:

10 Hazel: One number difference ... do you think that will always happen when we do this ... ?

Faye readily agrees, but Hazel, perhaps realising that she has not probed but has 'led the witness' seems to want to give them more of an option to disagree.

12 Hazel: What makes you think that? Just 'cos I asked it ... or ...?

Donna gives hedged agreement [14], and Hazel invites her [15] to account for her provisional belief.

14 Donna: I think so.

15 Hazel: Why?

Arguably this is a tough question - to account for a belief that one is not really committed to anyway. Donna's justification [16] is phenomenological rather than structural.

16 Donna: Well if um ... if it's after each other like ten, eleven, twelve ... um ... it will be one more because it's one more going up.

It is the basis of a subsequent expansive generalisation at the beginning of Episode 2.

62 Hazel: Okay. Right, what would happen if you had numbers that jumped up in two instead of one, so you had ten, twelve and fourteen?

63 Faye: I think the answer is a two number difference. So two.

64 Donna: Yeah, yeah. So do I.

The substantive proposition in [63] - that there is a two number difference - is, in fact, false. By prefacing it with an epistemic hedge, Faye marks her utterance as a conjecture and withholds commitment to it.

Returning to Episode 2: Hazel encourages the children to try out two more examples with three consecutive integers. They obtain a difference of 1 in each case and Faye [27] affirms her belief (unhedged) that, as Hazel puts it [26], "that will always happen".

26 Hazel: Do you think that will always happen then?

27 Faye: Yes.

28 Hazel: How can you say for certain 'cos you've only tried out three examples?

When pressed by Hazel to account for her belief [33], Faye attempts a start ("Because ... ") and then backs off [34]:

33 Hazel: ... why do you think that for certain?

34 Faye: Because ... well, I don't know for certain but I think ... 'cos the numbers that we've done are quite close to the first ...

Abandoning her attempt to respond to Hazel's request for an explanation, Faye's "well" [34] is a maxim hedge; she realises that her reply will not be fully cooperative. She cannot sustain the suggestion that she is certain, and is obliged to refuse Hazel's request. Her reply (a second part pair) is dispreferred in relation to Hazel's first part, and is marked by hesitation (Chapter 5). Indeed, her subsequent "the numbers that we've done are quite close to the first ..." is vague, hedged with an adaptor ('quite') and violates the maxims of Quantity and Manner. Her "well" suggests that Faye had foreseen the inadequacy of her explanation.

Donna offers a brief diversion:

35 Donna: I don't think it will happen if you do like eleven, fourteen, twenty-two.

36 Hazel: But you're talking about the one that ... if you always have a set of three consecutive numbers will it work?

Her "like eleven, fourteen, twenty-two" is a delightful example of a vague *generality* (in the sense of Peirce, 1934: see Chapter 2). It is the interpreter's task to determine what it points to, what is included by it. It is difficult to judge how Hazel interprets it, except that she takes it to exclude "three consecutive numbers" - and perhaps this is precisely what Donna intended to convey through her example. Evidently 'consecutive' is a useful but neglected item in the mathematical lexicon.

As in my conversations with Susie (Chapter 4), the pronoun 'it' is deictic in [35] and [36], anaphoric in fact, and co-referential with an earlier demonstrative pronoun 'that':

26 Hazel: Do you think that will always happen then?

Again, 'it' [35, 36] (and 'that' [26]) has no conventional name, but Hazel, Donna and Faye tacitly understand its deictic referent - a proposition which Hazel has come closest to articulating, much earlier in the conversation.

10 Hazel: One number difference ... do you think that will always happen when we do this where we've got three consecutive numbers and we multiply the two end ones ... and then in the middle?

Faye brings the discussion back on course with a request for a crucial experiment (Balacheff, 1988: see Chapter 1).

38 Faye: I'd like to try it out in the hundreds.

Donna's choice for the experiment seems to be guided by Hazel:

39 Hazel: [to Donna] You want one difference between each of those. If you're going to start with a hundred you could have a hundred and one, a hundred and one and a hundred and two. Would you like a calculator ...?

Faye's independent choice of "any old" set (Hardy, 1940, p. 106) of three consecutive integers - 110, 111, 112 - becomes apparent later [60]:

51 Faye: I still get one number different.

52 Hazel: So that ... so do you ... will it always work d'you think?

53 Faye: Yeah ... I think.

54 Hazel: How can you be sure?

55 Donna: Umm

56 Faye: [laughing] Well ...

57 Hazel: Are you sure?

58 Faye: Well not really, but ...

59 Donna: Quite yeah.

60 Faye: I think so. Yeah quite sure. Because it has worked because we've done ten, eleven ... Well I've done ten, eleven, twelve, nine, ten, eleven which are quite similar and then I've jumped to, um, um ... a hundred and ten, a hundred and eleven, and a hundred and twelve. It's quite a big difference. So yeah?

61 Donna: Yeah so do I.

By this stage Hazel seems reluctant [52] to influence their commitment to the generalisation (the 'it' that 'always works'). Faye's intellectual honesty is very evident here. Her crucial experiment [60] provides another (presumably weighty) confirming instance of the generalisation [51] yet her assent to it is still hedged, partial [53]. One senses that Hazel has created, or nurtured, a Zone of Conjectural Neutrality (ZCN, Chapter 5) in which Faye understands that it is the conjecture ('it always works') which is on trial, not her. She is free to believe or to doubt. Nevertheless, her 'well's [56, 58] indicate dispreferred turns; she senses, perhaps, that it would be easier if she agreed, that agreement would better respect Hazel's positive face wants - for Hazel would gain satisfaction from Faye's coming-to-know. At the end of Episode 1 she goes some way towards agreement [60], affirming in the end that she is "quite sure" i.e. even more sure than might be expected, the Adaptor "quite" making things "less fuzzy" (Lakoff, 1973, p. 471). She proceeds [60] to reflect in detail on the variety of evidence which she has assembled, to account for her willingness to make the enthymematic leap into the unknown. Donna is apparently something of a passenger compared with Faye.

I conclude this selective study of Hazel's transcript by considering Episode 4, which opens as Hazel invites the two girls to "recap" on their generalisations to date:

121 Hazel: Okay. So if you recap. Would you like to start from the beginning and tell me what the difference is when you've got a jump of one, what the difference is in the answers. When you've got a jump of two what the difference is in the answers.

122 Faye: Okay when you've got a jump of one the difference in the answers is one.

123 Donna: When you've got a jump of two the answer ... the difference in the answer is four.

124 Faye: And when you've got a jump of three the difference in the answer is nine.

The children observe that each difference is a square. [125-129] Next, Hazel invites a prediction:

130 Hazel: Can you tell me what the difference in the answers of the two sums that, the two multiplications you're doing would be when you have a difference of four between each number?

131 Donna: Twenty ... twenty-six

132 Hazel: You think twenty-six Donna. What do you think Faye?

Hazel's response [132] to Donna's prediction (incorrect as it happens) is of interest to me since I observed the same linguistic behaviour in myself (Chapter 5) - the use of an attribution Shield to sustain the involvement of the other child. In this case Donna corrects herself, and Faye agrees -

133 Donna: [interrupts] no sixteen.

134 Faye: Sixteen, yeah sixteen.

Donna's error [131] was perhaps computational ($4^2 = 26$) rather than algebraic (the difference is 4^2).

Next, Hazel invites predictions for differences of 5 and for 6. Donna and Faye give responses of 25 and 36. Hazel moves to the expansive generalisation [146- 155]

156 Hazel: What's the pattern then? Can you sort of explain the pattern for me?

157 Faye: Okay, if the difference between the numbers you have to begin with ... um ... is, if you times that by, if you multiply that by itself it will make the difference between the two answers that you get. Yep?

158 Hazel: You agree?

159 Donna: Yeh.

160 Hazel: Right, well done.

The FTA inherent in the request for "the pattern" [156] is softened by the modal form ("Can you?") and the hedged performative ("sort of explain"). Faye obliges with an account [157] which is characteristically extended, indicating a confident exposition of secure knowledge. It is hesitant only because she begins by reifying a variable - "the difference between the numbers you have to begin with" - but she must then cope with the burden of the English language, as opposed to symbolic algebra, to say what she wants to say about a function of that variable.

On a number of occasions (e.g. [156] and notably [83]), Hazel asks "why?" and requests "explain" with regard to generalities. For the most part these questions and requests seem to require *descriptions* of regularities rather than fundamental accounts of their causes. These two girls might well respond intellectually to some generic example or geometric model of the situation in Episode 1. But taken as a whole, the transcript is pure delight, a fine example of quasi-empirical teaching and learning, supported by a skilful and sensitive teacher.

CASE 2: ANN

Ann is an experienced primary school teacher. In the transcript (IRG2) she is talking with Charlie, a ten-year-old boy in her class. Charlie's mathematical attainment is judged by Ann to be below average for his age.

The conversation is essentially an exploration of residue classes modulo n. Charlie will share sweets among people to see how many are left over. Ann has prepared a 'table' on squared paper: she has written 2, 3, ... 12 along the top (for sweets) and down the left-hand edge (for people).

Ann begins by asking Charlie to share 2, 3, 4, 5 sweets among two people (herself and Charlie), and explains how to enter the number "left over" in the table. Her instructions are indirect speech acts, presented in modal interrogatory form:

9 Ann: Can you put one in that square because you have three sweets and two people so you have one left over. Can you share these out now? Four sweets. Are there any left over?

10 Charlie: No.

Charlie has done the 'practical work' sharing out up to five sweets, and has entered 0101 in the first row. He spontaneously predicts the next two (perhaps he lacks his teacher's patience?):

16 Charlie: This is probably going to be nought, one, nought, one, nought, one .

17 Ann: Think it is?

18 Charlie: Yes.

He shields his prediction [16]; Ann picks up the hedge and asks how confident he is - her 'think' [17] is more root than epistemic. In fact, Charlie is sufficiently confident [20, 22] to extend the table without recourse to further sharing of sweets:

19 Ann: Would you like to put it in before I give any more sweets out?

20 Charlie: Yes.

Evidently Charlie would "like to", but Ann is uneasy:

21 Ann: Are you that confident?

22 Charlie: Yes.

23 Ann: Go on then. Put them in. [Pause for Charlie to put in the numbers]

Ann's next instruction is thinly veiled by conventional indirectness:

23 Ann: Shall we check to make sure?

The "we" [23] expresses the teacher's solidarity with the child in the activity, and compounds the impossibility of Charlie's refusing the 'invitation' to "make sure". Refusal is clearly dispreferred and would offend the teacher's face wants - both positive (she is a partner in the activity and is suggesting a check) and negative (she is the teacher, he the pupil). Charlie checks his prediction, and Ann moves on to three people:

23 Ann: This time there are three people. Three people, if I get two sweets, can they have one each?

24 Charlie: Yes.

25 Ann: There are three of us remember.

26 Charlie: I mean, no.

These four turns, frozen in text, illustrate how transcripts can be used to reflect on classroom interaction and to develop questioning styles. Ann's question [23] is presented as a binary option - "can they?", yes or no?. Her response to his answer is neither confirming nor neutral, and leaves him in no doubt that he is being corrected. Naturally, he changes his answer. There is no evidence in the transcript as to whether or not he changes 'his mind'.

Ann, who donated the transcript, might use it to consider alternative ways to [23] of formulating the question. A number of possibilities come to mind - why the "I", which requires Charlie to put himself in her position?; who are the three people (a ruler is adopted later)? But in particular, it may be more effective in achieving intellectual participation from Charlie if it is not presented to him as a two-way choice, but in a more open ("Wh-?") form such as "What would happen if you ... ", or indirect variants such as "What do you think would happen if you ... ". Such a form would also be likely to elicit a more extended reply, with greater potential for insights into Charlie's personal construction of the situation. No entries are made in the table below the leading diagonal (i.e. one sweet each, none over), yet it is worth considering the fact that there are sweets 'left over' when there are fewer sweets than people, and that the corresponding entries would maintain the cyclic regularity of each row. The question [23] "Can they have one each?" focuses on the quotient (which is irrelevant to the modular regularity) rather than the remainder. As Paley observes (1981, p. 218), "The tape recorder trains the teacher not the child".

Under Ann's guidance, then, Charlie enters 0, 1, 2 in the row for three people. Again he spontaneously suggests (with the same Shield "probably") an extrapolative extension of the data, but this time he seems to be caught up in the rhythm of the language more than the logic of the situation:

38 Charlie: Probably going to go three, four, five, six, seven, eight, nine, ten.

This time Ann patiently encourages him (with the conspiratorial "Let's") to test his prediction:

39 Ann: Let's try it and see. [Counted out 6 sweets.] How many left over?

40 Charlie: None.

41 Ann: [Counted out 7 sweets]

42 Charlie: One.

43 Ann: Fine. [Counted out 8 sweets]

44 Charlie: Two.

Now she invites him to generalise:

45 Ann: Right. Now can you tell me how the numbers are going to go?

46 Charlie: Nought, one, two, nought, one, two, nought, one, two.

The practical work with the sweets is emphasised (laboured?) as [47-50] Charlie is directly instructed to "Share out and see" for 9 sweets. He counts out the 10 sweets without attempting a prediction; suspecting, perhaps, that he will have to do it in the end anyway. The only sense-making that is legitimised is in terms of the algorithmic manipulation (sharing) of the embodiment.

47 Ann: What's the next number going to be?

48 Charlie: Nought.

49 Ann: Share out and see. Well done! What about the next number? [Pause while Charlie counts out sweets.]

50 Charlie: One.

There is evidence in the transcript that, despite the practical work, Ann is directing Charlie's thinking towards the patterns of numbers on the table rather than the sweet-sharing. Is the 'train spotter's paradise' in sight? (Hewitt, 1992)

55 Ann: Right. Look at that and can you tell me what is going to happen on the next line? [*sharing between four people*]

56 Charlie: Mmm. I think we will get a three instead of a nought. [*he is apparently referring to the second zero in 012012, since he writes 012301230 below*]

And again, later:

63 Ann: We have five people now. Where are you going to start on our table?

64 Charlie: There. [*at (5,5)*]

65 Ann: What do you think the pattern is going to be in the numbers?

66 Charlie: Two, three, one, nought, two, nought, one I think.

67 Ann: Write it down there to remind you. Why do you think it's going to be that?

68 Charlie: Could be nought, one, two, three, four.

69 Ann: Put that number down as well. So could be nought one two three four. Let's try and see.

Charlie hedges [66] or modalises [68] his pattern predictions, which are still insecure [66]. Ann's enquiry in [67] is the first of only two 'why?' questions in the whole transcript, and this one causes Charlie to self-correct.

Ann's use of pronouns frequently associates herself with Charlie's progress in quite a personal way:

77 Ann: All right. You do that line **for me**. Well done.

79 Ann: **We** put a nought when there is nothing left over, didn't **we**?

The 'we' traps Charlie into complicity (Chapter 4) with Ann. At the very end, Ann assesses, with further reference to 'me' and now 'us', whether Charlie has understood - or at least remembered.

88 Ann: So can you tell **me** what the nought means? What is it telling **us**?

89 Charlie: That, um, you can share them out and have the right amount.

I suggest that Charlie's 'you' is not, however, addressing Ann (recall, from Chapter 4, that pupils rarely address their teachers directly in mathematical discourse), but an indication that he is articulating a generalisation, that he can be detached about the significance of the zeros.

The general purpose of the activity is very nice, very significant - the least positive remainder is less than the divisor. If Charlie stays the mathematical course, he may realise in ten years time that he has been investigating the Division Algorithm! I suspect that relatively few primary teachers would realise the potential of this activity, nor, perhaps, predict how demanding it would prove for Charlie. On the evidence of the transcript, Ann has judged that Charlie will need a good deal of guidance and practical support as he works through the investigation. The same evidence suggests that Ann's attention is on his practical performance rather than his cognitive structuring and his propositional attitude - the way he construes the patterns in relation to the sweet-sharing and the strength or fragility of his conviction. There is no sign of examination of conjectures in the ZCN. She seems to find it hard to release him from the practical task, from dependence on her as teacher and even from obligation to her as 'partner'.

CASE 3: JUDITH

Judith is a 'newly-qualified teacher' in an 11-16 secondary school. The transcript she has prepared (IRG5) is the most extensive of those donated by the IRG members - about 3000 words. In the transcript she is talking with Allan, an "average" Year 9 (age 13-14) pupil.

The conversation (which must have lasted about 45 minutes) concerns the problem of drawing line segments between pairs of dots in a plane array of some sort (usually regular). What is the least number of segments necessary so that they form a continuous line connecting all of the points?

It falls into five episodes:

Episode 1: Investigation of a 3x3 square array [IRG5:1-49]

Episode 2: Prediction and verification for a 4x4 array: expansive generalisation for square arrays to include 3x3 as a special case [IRG5:50-93]

Episode 3: Prediction, verification and expansive generalisation for rectangular arrays, including squares as special cases [IRG5:94-151]

Episode 4: Consideration of triangular arrays: reconstructive generalisation (Harel and Tall, 1991: see Chapter 1) based on number of dots in the array which includes the three generalisations arrived at inductively in the previous episodes as special cases [IRG5:151-160]

Episode 5: Search for regularity in the sequence (of numbers of line segments) for triangular arrays - arithmetic sequence of differences [IRG5:160-264]

The episodic overview immediately reveals the richness and cognitive complexity of the layers of generalisation that are built up in the first four episodes. The first of these begins:

1	Judith:	OK, Now. Here is the investigation, so you can read it. Draw a three by three dot grid. Start anywhere you like. Draw a continuous line that goes to every dot. Yeah?

Notice that we have entered the culture of the post-Cockcroft secondary school - 'doing an investigation' (Love, 1988, p. 250). In contrast to the examples from primary schools, the investigation is presented to Allan in a written format. This may reflect Judith's inexperience (it feels 'safer'), but in any case it is how investigation 'starters' *are* normally presented to secondary school pupils, for GCSE coursework and the like.

Judith makes reference to the rituals of 'doing an investigation' culture on several occasions in the interview:

88	Judith:	So if you were doing an investigation what would you write down for me?
160	Judith:	So ... what would you write down if you were doing an investigation?
130	Judith:	... if you did it three, four, five, six it might be easier to see patterns, do you not think? Do you do tables when you do investigations ... ?

Judith exploits the familiarity of the investigation 'write up' to encourage Allan [88] to articulate his thoughts, and with some success:

89	Allan:	I'd write ... that the pattern is ... if you ti-, times both, if you square the side, the side, and um, you minus one, you'd be th', the amount of dots you-, if you went round the dots it'd be the same answer.

Allan's approach to speech as imagined writing seems to assist him in assembling his thoughts, and he gives (with a few false starts) a relatively formal account of a generalisation. Presumably oral 'reporting back' (Pimm, 1992, pp. 68-72) is not part of the practice of school investigations with which Judith expects Allan to be familiar, since it would otherwise be a more natural point of reference for him. Writing is, on the whole, more dense, more formally structured than speech (Perera, 1990; Brown and Yule, 1983, p. 15), and Allan still has need of the informal generaliser 'you' to formulate his rule.

To return to Episode 1: Allan is quite relaxed and competent in the use of epistemic hedges:

11	Judith:	OK, Do you think you're going to be able to do it in less than ... that? ... nine?
12	Allan:	Maybe, yeah.
12	Judith:	Maybe.
13	Allan:	Maybe.
14	Judith:	OK, so you're not ...
15	Allan:	Not positive, but I am ...
16	Judith:	OK [pause] Top right [pause] Are you thinking about where to go next?

Judith gently echoes [13] and explores [15] his uncertainty; her interruption [17] relieves the tension but cuts off the flow of data from the informant. Before long he has 'done it' with eight line segments (starting with a corner dot), but he remains tentative [24, 30]:

24	Allan:	Still eight again so probably the most is eight.
25	Judith:	You mean the least?
26	Allan:	Yeah, the least, sorry.
27	Judith:	That's alright.
28	Allan:	Yeah.
29	Judith:	OK, so do you think starting not in the corner could get you eight as well?
30	Allan:	Um, possibly, yeah. [pause]

His doubt appears at first to be well-founded:

40	Allan:	So it don't work from the middle at all, really, because it's uh, because you have to go in and out again.

41 Judith: D'you think? Are you sure?

42 Allan: Because what you have to do, you have to go, from the middle you have to go all the way round and you go into one and come back out again, but if you do it from, like we 'ave did from exactly, exactly in the middle.

Allan's utterance [40] provides examples of procedural deixis ('it don't work') and the generaliser 'you'. I would argue that his 'really' [see also IRG5:56 and 217] is a hedge on his claim that 'it don't work', much the same as 'basically' (Chapter 5). Judith tests his claim [41] in the ZCN, and Allan responds with an extended, if somewhat incoherent account, making heavy use of the generaliser 'you'. I am reminded of one particular moment when Susie 'explained' to me with a long, rambling and quite incomprehensible speech [S1:30] to which, completely lost, I weakly responded "So what did you do next?" in the hope of some behavioural clue. Judith's next turn [42] has some of that 'lost' quality:

43 Judith: Try that then. [pause, Allan draws a route from the middle]

In fact, Allan finds the counter-example to his own argument:

44 Allan: Yeah, made eight as well from the middle.

Judith now needs to know quite where Allan stands at this stage. She uses the 'if you were doing an investigation' strategy to ask for a summary progress report:

45 Judith: OK. So what do you think? [pause] So if I wasn't talking to you and you were just doing the investigation yourself, what would you be thinking?

46 Allan: I'd be thinking that the most possible um way uh of getting, of getting, getting the least is that it only started from the middle going right round, going right round or going, going from one corner but you can't do it in the middle, middle between two corners.

47 Judith: OK.

48 Allan: And also the least is eight.

She invites Allan (ushering in Episode 2) to determine where the investigation will go next -

49 Judith: Alright then, so what're you going to do now?

50 Allan: I'll try a, um, four by four grid.

- but once he has decided, she explores - with an indirect request [51] for a prediction - whether and how his thinking is becoming structured, whether he has any generalised overview of the problem.

51 Judith: Right. Can you make any predictions before you start?

The indirectness softens the force of the FTA and his hedged prediction [54] suggests that he may have formed a generalisation from the single 3x3 instance.

54 Allan: The maximum will probably be, er, the least'll probably be 'bout fifteen.

The epistemic adverbial Shield 'probably' is reinforced with the Approximator-hedge '(a)bout'. This is an interesting use of an epistemic Rounder (Chapter 5), repeated later by Allan (below) in [73]. Incidentally, it is plausible, but unlikely in my view, that 'fifteen' is being used as a 'round number' (Channell, 1994, pp. 87-89), and an Approximator in its own right. Judith wants to explore the thinking behind the prediction:

68 Judith: So why did you predict fifteen?

69 Allan: Uh, because I thought there might be a pattern between ... if there was um a certain amount of, um ... if it's three by three say ...

70 Judith: Uh-hum.

71 Allan: if you ti-, three times three is actually nine.

72 Judith: Uh-hum.

To begin with, Allan is struggling; perhaps the linguistic struggle is the manifestation of a metacognitive struggling to recover or construct the reason for his prediction. His overture ("I thought there might be") is uncommitted. At first he finds it helpful to illustrate [69, 71] with the 3x3 example, rather than to articulate the generality. Judith waits; her interventions [70, 72] are absolutely minimal. She says just enough to assure Allan that she is listening, like a counsellor listening to a client. She is soon rewarded with an explanatory outpouring, punctuated with the impersonal 'you':

73 Allan: But as, if you went round all the dots, it would only come to about, if you did it once it would come to on-, uh less than nine, 'n' you got, uh, because, because there's o-, there's only ... 'cause you only have, y- ... you can miss out a line exactly, 'cause y- you can miss out a gap, c- 'cause you um, y'd 'ave to go all the way round the whole dots.

So what, asks Judith [74], are the significant generalising features of this 3x3 example? What does the generalisation look like? [76]

74 Judith: OK ... So why did that make you say fifteen?.

75 Allan: Because uh, f- for the same reason, 'cause if you um w- tried to go round the whole all the dots you'd get sixteen but if you just did it once all the way round the dots but missing out gaps you'd still come to uh, you just minus one basically and just ...

76 Judith: So what would happen in some other squares?

77 Allan: Probably if you minus one from the s-, if you square the number you'd probably find that if it was actually, if you minus one from that you'd probably find that that would be the answer to the ...

The most refined 'algebraic' account that Allan is able to develop [77] is divorced from the dots and the gaps. It is an algorithm, albeit a highly ("probably") tentative one. I'm still here, says Judith [78]:

78 Judith: OK.

79 Allan: ... to how many dots there are, to how many times you 'ave to go round the dots

80 Judith: OK. Now do you want to try one, or are you certain of that?

Judith's question [80] seems to present Allan with a genuine option. It really is a genuine question rather than an indirect instruction. The 'or' is explicit. There is no obvious preferred response. Contrast it with Ann's earlier (Case 2):

IRG2:23 Ann: Shall we check to make sure?

In fact, Allan goes on to say that he is not certain, and tries out the 5x5 array.

I have examined less than one third of the transcript; 15 minutes talk perhaps. My purpose in studying these transcripts was to look in them for the linguistic features that I had identified in my own conversations with pupils, and for some validation of the conclusions I had reached about the pragmatic function of vague language. It was not my intention to say what was 'good' or 'bad' about the teaching, but the analysis inevitably puts into sharp relief those aspects of practice which support or negate a conjecturing atmosphere - assuming that that is what is wanted. In that respect, Judith's instincts are remarkably true.

CASE 4: RACHEL

Rachel is in her third year of teaching. She works at a sixth form college with a strong academic reputation. Rachel interviewed two pairs of 18-year-old students, all following an Advanced Supplementary (AS) course in mathematics. [Note 7.1]

The students were presented with the following investigation (in written form).

> You need to climb a staircase with *n* steps. You are allowed to go up the steps taking either 1 or 2 steps at a time. In how many ways can you go up *n* steps? As an extension consider being able to take 1, 2 or 3 steps.

in the first transcript supplied by Rachel (IRG6A), the two informants, Juliette and Di, are described as bright students who work well together in class. In the event, both gained A grades in the AS examination. Rachel begins by checking that the task is clear.

IRG6A:1 Rachel: So, do you think you understand what it means?

The drawn-out form of her question ("So, do you think ...") could indicate that she is conscious that she might be thought to be patronising these able students. Juliette clarifies the task, seeking confirmation with the tag-question "can't you?" [3, 7]

3	Juliette:	So you can have a combination of ones and twos, can't you?
4	Di:	It's going to take n steps or n over two steps.
5	Juliette:	Or a combination. You could look at ...
6	Di:	You could take n as being one. [writing table] x,y
7	Juliette:	If n is two, you could do either two steps or one, can't you? n equals three ... [writing down 2 1, 1 2] Does it count if you do two and one and one and two?
8	Rachel:	Yes, they're different.

In [3, 5] and throughout the interview, Juliette freely uses 'you' for generalisation and/or detachment. Di quickly goes into 'investigation mode' with a table of (x, y) values. Juliette clarifies the rules on 'sameness' ("Does it count ..?") and Rachel adjudicates [8]. "Does it count?" seems to be a standard legitimation enquiry, just as "It works" is a standard procedural generalisation. By [10] Di has made a (false) prediction of four ways for four steps:

10	Di:	Yeah, [writing out an (x, y) table] so you have one, one; two, two; three, three; so four is four.

Her prediction is not marked in any way as regards uncertainty. What is the interviewer - also, here, the teacher - to do?

11	Rachel:	I think maybe you need a few more before you can generalise.

The double plausibility Shield [11] is, of course, play-acting on Rachel's part. Rachel wrote a reflective account of her two interviews after she had transcribed them (Williams, 1995) and comments:

> I feel I have to interrupt and prompt her to consider a few more cases. In retrospect, perhaps I should have waited to see if Juliette did that.

Soon, the two students are working on the case n=5:

18 Juliette: ... Two, one, one, one, one, two, one, one, one, one two, one [more writing]. That's it. So, we've got one, two, three, five, eight and now you're going to get twelve.

19 Rachel: Why?

20 Juliette: It's a series. You add one. You had one, that's that's ... you had one. Oh, I don't know.

21 Rachel: Yes, yes, you add one.

22 Juliette: You add one, then you add two, then you add four, the interval between. No, that's right now. It's something to do with the series it goes up with.

23 Rachel: Mmm.

[I wonder whether 'had' in [20] should be transcribed as 'add']

This time it is Juliette who is misled to a faulty prediction (12 ways for 5 steps) by an alternative regularity [22] in the first few terms. Again, her prediction [18] is unhedged. She freely decentres with 'you' [18, 20, 22]. Note that "we've" got one, three, five etc. (actual data) whereas "you're" going to get twelve (prediction). Finally "I" don't know (a personal epistemic state). Rachel asks for an explanation [19], which is taken to be a request for an account *of* the perceived regularity [20]. Rachel enthusiastically encourages [21] Juliette's faltering start which seems to be based on addition. The continuation [22] perpetuates the faulty prediction, but is not committed to it. Rachel's response [23] is minimal (c.f. Judith, IRG5:70, 72]. She writes (Williams, 1995):

> Juliette has the right sort of idea [22] and says "It's something to do with ... " but she will not commit herself. I am trying hard not to interfere and biting my tongue with "Mmm".

In the end Di is surprised to find more ways than Juliette had predicted [31] and Rachel can restrain herself no longer:

31 Di: [counting all the combinations] ten, eleven, twelve, thirteen [delighted/ puzzled]

32 Rachel: That's all right.

33 Juliette: [puzzled] That's O.K?

Rachel's comment:

> I had to confirm that she was correct [32], I couldn't bear the uncertainty and wanted them to know they had got to the correct number of ways. Looking back it would have been better to let them sort it out. (*ibid.*)

Rachel brings out another affective dimension in the conjecturing atmosphere. The pupil is required to take risks, but the teacher may have to "bear the uncertainty" when she judges that the pupil must resolve it him/herself. That is not to say that the teacher cannot participate in the ZCN, but her/his role may be best restricted to a light, indirect, linguistic scaffolding.

The two students are hooked onto familiar sequences (arithmetic, geometric) and are somewhat inflexible in their search for regularity in the (Fibonacci) sequence 1, 2, 3, 5, 8, 13 ... In the end, Rachel finds it hard to allow them to flounder:

35 Juliette: Times two, no, no direct ... no common difference no common ratio.

36 Rachel: But when you were doing your thing of adding on each time it worked. Well ...

> I sense that I am hesitant in letting them search too long for a solution and that I assume they want to get to the answer quickly. I am in some way anxious that it is taking them a long time, but in interrupting, I interrupt interpreting conversations they could have in getting to the answer ...
>
> I was anxious that it should work and I think that bright students should be able to get to the answer without too much difficulty. I did have expectations of the students ...
>
> I felt that their knowledge got in the way of their intuition. They felt that the question should fit into an arithmetic or geometric progression type question.
>
> I felt that I intervened too quickly and didn't let them struggle enough. Perhaps I was more nervous of the tape than they were - this was my first interview. (*ibid.*)

In her second interview (IRG6B), Rachel discussed the same problem with Clare and John. Rachel judged Clare to be the more able of the two students; she later gained an A in the AS examination, John a C. She takes the lead in the conversation, and soon arrives at and articulates the Fibonacci 'rule'.

14 Clare: So that gives us two, five, eight. You just add on the previous number.

Prompted by Rachel, she predicts and verifies the next term, 13.

15 Rachel: Do you want to check it then? What would you predict the next one would be?

16 Clare: Hmm, thirteen.

There is no evidence of an attempt to account for the observed and confirmed regularity, and no enquiry as to John's commitment to it. They move on:

19 Rachel: O.K. So try this one then. If you could take one or two or three steps.

Before long, Clare articulates and revises a provisional generalisation, based on the first three terms of the sequence :

25 Clare: Do the next one then, four. It's squared, so it'll be nine. Probably. No, that's double two, you double then ...

Clare is resourceful in generating conjectures for 'the pattern' on the basis of the data available; thus, with the terms 1, 2, 4, 7 she ventures:

30 Clare: All right then, you add up all the numbers before.

31 John: Seven.

32 Clare: So the next one will be fourteen, maybe? Three, seven, fourteen. Do you think maybe you just add, mm, no, it doesn't work there. O.K.

The provisional status of the generalisation [30] only becomes apparent in the Shields [32] that she uses when she applies the rule to predict the next term (1+2+4+7=14). Very soon she modifies the rule - it is not clear from the transcript whether they find that there are in fact 13 ways for 5 steps:

36 Clare: Add the previous three, because when you had two steps you added the previous two numbers and then you have three steps, you add up the previous three numbers.

This generalisation is expansive insofar as Clare relates it to the 'rule' for "two steps". The "because" refers to the *form* of the generalisation rather than to any underlying reason. There is no linguistic sign of uncertainty at this point.

Perhaps because of Rachel's recognition of the face wants of her students, and her desire for and expectation of their success, she seems to have difficulty in allowing them to struggle. There is a hint of bravado in the three female students' approach to the problem, without much linguistic evidence of uncertainty. Perhaps, without being pressed to consider proving their inductive conjectures, the problem is not in fact much of a threat to their mathematical self-esteem. John's contribution to the conversation is marginal, dealing with minor clerical and arithmetic matters. His turns are brief, and give little information about his propositional attitude.

CASE 5: SUE

Sue, an experienced teacher, works with the reception class in a primary school. She provided transcripts (IRG1A-D) of four short interviews with four-year-old children.

A: Rebecca, aged 4:8

B: Jane, aged 4:8

C: Anna, aged 4:3

D: Jason, aged 4:11

The task in each case was as follows.

Sue placed five plastic 'people' on the floor, and asked the child how many people there were. Next, she asked the child how many more people would be needed to make ten. In two cases she then asked how many more would be needed to make twenty.

I shall consider the interviews together with reference to some common features. Some of these confirm my observations about Year R children (under 'Initiation') in Chapter 6.

(i) Adult behaviour and questions are taken at face value, the child is naively cooperative, and simply acknowledges her/his ignorance [D10], error [C4] or uncertainty [C8, 10] for what it is:

D9	Sue:	How many more do we need to make twenty?
D10	Jason:	Umm, don't know.
C2	Anna:	[pause] Umm ahh one, two, three, four.
C3	Sue:	Four, you think?
C4	Anna:	No, one, two, three, four, five.

C8	Anna:	One, two,three, four, five, six, seven, eight, umm, nine, I think we need nine more.
C9	Sue:	You need nine more?
C10	Anna:	I think so.

Like Jason, Jane knows her limitations, and it does not appear [B10] to be a face-threatening issue for her.

B9	Sue:	How many more do you think we'd need to make twenty?
B10	Jane:	I don't know. I can't count up to twenty. Only my sister can.

(ii) Nevertheless, Sue is frequently indirect in her instructions to the children. In the later meetings of the IRG it was very apparent that Sue wanted to create and sustain a conjecturing atmosphere in her classroom, one in which the children knew it is 'alright to be wrong'. Thus, her first question is direct in every case, presumably because it is the bread-and-butter of the reception class -

C1 Sue: Right Anna, how many people are on the floor?

- whereas the next question, which she suspects (with good cause) will challenge them is twice endowed with a Shield, and always with 'we' for solidarity:

C5 Sue: How many more people do you think we need to make ten people?

Her first question [C1 and see B1and D1 below] is in every case heralded with an utterance initiator "Right" (three times) or "OK" (once). This interests me because I know (from transcripts) that I use 'right' a great deal. On reflection, I suggest that 'right' and 'OK' have three distinct pedagogic functions. The first (that identified above) is to indicate boundary points in a 'lesson'; such markers (which also include 'now', 'now then' and 'well') are called 'frames' by Sinclair and Coulthard (1975, p. 22), who observe that they initiate a teaching 'unit' in which they are typically followed by a meta-statement of some kind. For example:

I:7.1 Right, we're going to start with a quiz today.

The second function is a minimal interjection into a pupil's account of something (an explanation, for example) or attempt to formulate such an account. It assures the main speaker (the pupil) of the listener's attention. See particularly Case 3 and Case 8 for examples. The third function, as a statement-tag, is to seek approval (pupil use) or to seek assurance of agreement or comprehension (see also Note 7.2)

In the two cases where the child incorrectly assesses the size of the initial set, Sue counters with the Attribution Shield "you think", which I have noted (Chapter 5) as a feature of my teacher-strategy. For example (to Rebecca):

A7 Sue: You think six. How many more people do you think we would need to make ten?

(iii) It is very clear that (see Chapter 6), for three of these four Year R children, a "How many?" question triggers a count. The reaction is almost Pavlovian:

B1 Sue: Right Jane, how many people are there ?

B2 Jane: One, two, three, four, five.

D1 Sue: O.K. Jason, how many people have we got here?

D2 Jason: One, two, three, four, five.

Examples of the same phenomenon are recorded elsewhere (e.g. Walkerdine, 1988, p.106). The response to the stimulus question "How many ..." is a recitation out loud of the 'standard number word sequence' (SNWS - Steffe *et al.*, 1993 p. 25) of 'numerons' or counting 'tags', but there is no external evidence that the child has mastered the cardinal principle (Chapter 6), that the last tag used is the cardinality of the set, in evidence of which

> [...] the child must be able to pull out the last numeron assigned and indicate that it represents the numerosity of the array. (Gelman and Gallistel, 1978, p. 80)

There may, of course, be prosodic features of the children's counts that "pull out" the last number in the recitation, but this is not indicated in the transcripts.

(iv) Lack of mastery of the cardinal principle may be related to the non-standard response of Jane and Anna to the "How many more?" question.

B3	Sue:	How many more people will we need to make ten?
B4	Jane:	[pause] Umm, six, seven, eight, nine, ten.
B5	Sue:	Right, do you think you could find some people to make ten?
B6	Jane:	Just yellow ?
B7	Sue:	I don't mind. You can have whatever colour you like.
B8	Jane:	One, two, three, four, five, six, seven, eight [pause to rummage] and a yellow one [puts down two more] ten.

Jane's response [B4] is to extend the recitation of the SNWS beyond the point she had reached ("five") to "ten". The purpose of Sue's contingent question [B5] seems to be to discover whether Jane will match the five numerons ("six" to "ten") with five people. In fact Jane legitimately interprets [B5] as a request to get a new set of people to "make ten".

Anna does offer a prediction - unhedged - [C6], adds about the right number to the original set of people, but suggests (Shielded) that the cardinality of the union is answer to the 'more' question.

C5	Sue:	How many more people do you think we need to make ten people?
C6	Anna:	Three.
C7	Sue:	Three? Do you want to find out? [Anna adds more people]. Now how many have you got now?

C8 Anna: One, two, three, four, five, six, seven, eight, umm, nine, I think we need nine more.

Similarly, Jason's response to "How many more do we need to make twenty" (IRG1D) is to count (with an unstable SNWS) to twenty.

Despite Sue's experience as a teacher of young children, these responses surprised her. She wrote (personal communication):

> I found I had made assumptions about their basic mathematical language. We have taken a great deal of trouble (we use Ginn reception maths teacher's book and ideas of our own) to teach the children what they will need [...] but many did not understand "more" for example.

The problem probably originates in the particular precise and situated meaning of 'more' and 'How many more?' in the social practice of school arithmetic (Walkerdine, 1988, pp. 22-27). Walkerdine notes that, in the child's home, 'more' is associated with "food regulatory practices" (p. 26), so that (for example) the opposite of 'more' is not 'less', but something like 'no more' or 'enough'.

Rebecca is something of an exception in these four interviews. She is alone in not being kick-started into a vocal count by the initial "How many?" question. She either estimates or internalises the SNWS (the transcript does not reveal which):

A1 Sue: Right Rebecca, how many yellow people are there here?

A2 Rebecca: Umm, six.

There is evidence that this is in fact an estimate, in that she twice (subsequently) asks Sue whether she should count [A6, 10] and, moreover, that she can count small sets accurately. She appears also to estimate how many more are needed to make ten.

A4 Rebecca: Umm, two more.

A5 Sue: Two more? Do you want to try that then?

Sue is correcting Rebecca by repetition of her answer (Drew, 1981 p. 252), inviting her self-correction. The transcript notes that Rebecca selects three people. Sue continues:

A5 Sue: Right, how many people are there now?

A6 Rebecca: Ten. Shall I count them and see?

A7 Sue: That's a good idea.

A8 Rebecca: One, two, three, four, five, six, seven ... [pause] shall I do some more along there?

Rebecca's seems to make more conventional sense of the "How many more?" question. On the other hand, Sue does not press her for an answer to it.

A10 Rebecca: Shall I count them and see again? [Sue smiles] One, two, three, four, five, six, seven, eight, nine and ten.

A11 Sue: Thank you Rebecca.

In later meetings of the IRG Sue has commented on her growing awareness of the need to use language which emphasises the autonomy of the young children whom she teaches. In effect, she wants to equip them with 'basic' arithmetic knowledge and competence, but wants them to retain responsibility for sense-making and validating their solutions to problems. Yet sense-making in school mathematics is not solely a matter of private interpretation within some absolute, secure reality of 'real' objects ('people' and the like); it is also one of of linguistic enculturation, of initiation to a discursive practice (Walkerdine, 1988, p. 128).

CASE 6: THE PUBLIC LECTURE

This case is not so much a conversation but an example of a ritual mathematical monologue - the lecture. I propose merely to set the scene, to present some of the data and typographically highlight some relevant features.

Each year, Christ's College, Cambridge, elects a Lady Margaret Fellow, a 'visiting scholar' whose academic expertise may be in any area. In return for the year's fellowship, the Lady Margaret Fellow must deliver one public lecture "for a general audience". Last year, unusually, the fellow was a mathematician - Professor Herbert B. Keller of the California Institute of Technology.

The title of his lecture, given on 2nd February 1994 was: "How many lattice points lie in a circle?". Essentially, the area of a convex region of the plane is roughly equal to the number of lattice points (of a square grid) which lie inside it. Therefore a first approximation to the answer to the question (the lecture title) is πR^2 where R is the radius of the circle. If the error $\varepsilon(R)$ in this approximation is of the order of R^θ (so that $\varepsilon(R)=kR^\theta$) for some θ, what bounds can be put on the exponent θ?

The data which follow were all spoken by Prof. Keller.

From the preamble:

> At the Isaac Newton Institute in July last year, Andrew Wiles reported on his **almost-proof** of Fermat's Last Theorem.
>
> Surface tension **sort of** holds the water drop together.
>
> **It turns out that** the differential equation is the same - a form of **hand-waving** - I could tell you but ...

Into the substance of the lecture:

From numerical studies, a sample with $R<1800$ **suggests that** $R^{1/4}$ is better than $R^{2/3}$

So this **suggests that** R around 10^8 is needed to show that $\theta<0.6$

Hedging is most apparent in Prof. Keller's necessarily unprepared answers to questions after the lecture proper:

Half is still a lower bound, but **it isn't so apparent that** 7/11 (due to Iwanec and Mazzecci 1988) is still an upper bound.

The envelope **suggests that** $q<0.575$

I don't think that he (G. H. Hardy, 1915) got the conjecture from looking at Bessel functions.

I think that's what led to the conjecture"

I don't think that's been observed yet.

We might be able to show that, by accumulating jumps, we could make the bound rigorous.

The significance of the vague language used here by this 'expert' mathematician is that it is not perceived as a deficiency, either or language or comprehension, but as an acquired expertise, enabling him in each assertion to be as precise as he chooses.

CASE 7: OPEN UNIVERSITY VIDEO

Debbie teaches in a primary school in Suffolk. She was one of a number of teachers filmed by the Open University production team for course EM236. By a nice coincidence, Debbie was also one of the seven teachers whom I had observed five years earlier in a study of the introduction of CAN (a calculator-aware component of the PrIME project) in Suffolk in 1986-87. At that time she was a newly-qualified teacher. When I interviewed her for the study, she considered that the investigative approach to teaching and learning, which was central to CAN "fits in comfortably with what I was doing before" (Rowland, 1994b, p. 34), although she recognised that she was giving children more thinking time, "waiting, prompting if necessary, but not telling them" (p. 35).

In the video Debbie is working with a group of six children whose age (unspecified) appears to be about 7 or 8. The children are considering the number of unit cubes in 'cross' formations made with cuisenaire rods.

Her questioning is frequently but not invariably indirect, for example:

EM1:1 Debbie: What sort of shape would you call that? Kathleen [C4]?

9 Debbie: Could you tell me what the ninth one would look like?

44 Debbie: Okay, can you tell me how many cubes you'd need then?

She invites the children to extrapolate the sequence of geometric formations [9] and to compute the numbers of unit cubes in them.

22 Debbie: The ninth cross you've made, how many little cubes would you need to make that?

26 Debbie: Michael agrees, right, how many little cubes to make the thirty-seventh one?

[22] exemplifies the tension between composing sentences with correct syntax ("How many little cubes would you need to make the ninth cross?") and the desire to present the item which is to be the chief focus of attention of the audience (here, the ninth cross) without delay in the sentence. [Note 7.3] This point is discussed further in Case 8. In this case the use of an anaphoric demonstrative pronoun "that" becomes necessary; the pronoun is co-referential with the object of the subordinate clause which has been highlighted at the beginning of the sentence.

In Chapter 5, I noted my own tendency to use attribution hedges (such as "Frances thinks that ...") to sustain the intellectual involvement of other children at particular moments in the conversation, particularly when one pupil has given "the answer" to a question. Debbie uses the same linguistic strategy, but more as a device to side-step evaluation of their answers and suggestions:

20 Debbie: Now you think that's the ninth one. If the box only had hundreds of these little white cubes in, how many would you need to use to make the second one?

[26] has the same quality, in that Debbie is the chairperson for the discussion, ensuring that the state of play is understood by all, the views of all are heard and considered:

38 Debbie: Is that the thirty-seventh one, Alex?

39 C3: It is.

40 Debbie: Alex doesn't look convinced.

The girl C5 displays competence in the use of marked language - hedges and modal forms - when the children are asked how many unit cubes there would be in the thirty-seventh cross. This is an object which they must conceptualise first and construct later as a conservative extrapolation of the smaller, more tangible formations.

26 Debbie: Michael agrees, right, how many little cubes to make the thirty-seventh one?

27 C?: Oh!

28 C5: More than a hundred I should think.

C5 seems to be offering a vague estimate of the kind of size she would expect, and the estimate is appropriately hedged. She gives further emphasis to the hypothetical nature (Coates, 1983, p. 5) of her estimate with the modal 'should'. The modal-hedge combination "I should think" is doubly cautious, and unusual for such a young child - it does not occur anywhere in the Make Ten data.

Later, the two girls mis-calculate the number of unit cubes; C5 indicates her lack of full commitment, again with an epistemic modal [46]:

46 C5: It might be ...

47 C5&6: A hundred and seventy-nine.

48 Debbie: How did you get that?

49 C5: Well ... [laying out a limb with three 10s] one, two, three ... [lays a 5] thirty-five [lays two 1s] thirty-six, thirty-seven.

C5's "Well" in [49] suggests a possible Quality hedge i.e. they didn't in fact get their (erroneous) answer by laying out the cross in Cuisenaire. When they do, C5 counts the tens, then the fives, then the twos, and correctly computes four thirty-sevens.

The nature of the learning here does not involve the children in making inductive conjectures or enthymematic leaps. The sequence of 'crosses' considered (at Debbie's suggestion) is interesting - not so much consecutive as recursive (2, f(2)=9, ff(2)=37, fff(2)=149). Debbie is focusing the children's attention on the geometric generalisation in the form of the crosses - four arms of equal length, with an additional unit at the centre. C1 gives evidence of having made this generalisation.

45 C1: Oohhh! ... four times thirty-seven, no ... yeah. Four times thirty-seven. Well, almost ...

54 C2: You didn't count the one in the middle.

55 C5: Oh yeah!

Paul Cobb and his collaborators provide a brief but pertinent discussion of the teacher's role in classroom conversations in which he or she participates, appearing inevitably as an authority figure, yet having come to that interaction with commitment to a constructivist theory of knowing.

> One feature of the teacher's active and demanding role is therefore to facilitate mathematical discussions between students while at the same time acting as a participant who can legitimise certain aspects of their mathematical activity and sanction others. In doing so, the teacher ideally provides a running commentary on the students' constructive activities from his or her vantage point as an accultured member of the wider community [...] in a communicative context that involves the explicit negotiation of mathematical meanings. (1992, p. 102)

A significant factor in the above teaching sequence (Case 7) is Debbie's avoidance of both legitimising and sanctioning, her refusal to assume the role of truth-assessor, insisting rather that the children take responsibility for the validity of their own solutions "which must occur in order to allow the construction of meaning" (Balacheff, 1990, p. 259).

CASE 8: JONATHAN

My last case study concerns Jonathan, who was an undergraduate mathematics/ education student. Half of the four-year course that he was following involves the academic study of mathematics, and one of his third-year options was a 'paper' in the Theory of Numbers. The paper is assessed by a three-hour examination and two 25-hour 'projects'. Students choose their projects from a menu of seven starting points, and are normally given two one-to-one 'supervisions' on each project. I tape-recorded supervisions with a number of students, and selected one (Appendix 3, NT4) with Jonathan for analysis on account of its mathematical and linguistic richness.

Jonathan came to see me at 9.15 on the last Monday morning of the Lent Term of 1995. The appointment had been made at the conclusion of our previous supervision meeting a week earlier. For this project, Jonathan had been working on the problem of finding the number of integer solutions of $x^2+y^2=n$ modulo a prime, p. At a previous supervision he had discussed the cases $n=0$, 1 with me, including a proof for the case $n=0$. Since then he had generated some data for $n>1$.

The conversation was about 45 minutes long; the transcript [NT4] can be separated into five episodes:

Episode 1: Recall of the previous supervision, including a sketch of the proof of a theorem about the number of solutions when $n=0$ [NT4:1-73]

Episode 2: Elaboration of the proof, guided by Tim [NT4:74-135]

Episode 3: Jonathan talks about the case $n\neq0$ [NT4:136-156]

Episode 4: Discussion of particular values of $n\neq0$; when n is a quadratic residues mod p; proof that the number of solutions is the same for all such n [NT4: 157-208]

Episode 5: Jonathan's proof that, for all $n\neq0$, there are $p+1$ solutions if $p\equiv3$ mod 4, and p-1 solutions otherwise [NT4: 209-240]

Much of the generalising, the forming of conjectures, had taken place at the previous supervision. The process most to the fore in this encounter is proof. Jonathan's ideas are mostly skeletal, in need of detail, and my role (as I perceive it) is to provide some scaffolding around the construction of the details.

First, I need to ascertain what the conjectures are, and what progress Jonathan has made with the proofs. There is discomfort in the transcript as Jonathan is submitted to cross examination. Indeed, a hallmark of the whole transcript is hesitation on the part of both speakers, both of whom frequently seem to find it difficult to 'spit it out'.

1	Jonathan:	Well, I had a bit of a bash this time with the theory.
2	Tim:	Right .. we're talking about x squared plus y squared equals n ...
3	Jonathan:	Yes
4	Tim:	Yes, yes ... um, can you ... did we discuss equals zero last time?
5	Jonathan:	It ... came up, yes.
6	Tim:	Right ... you'll forgive me, but I've discussed the same/
7	Jonathan:	/yeah (yes)/
8	Tim:	question with one or two people.
9	Jonathan:	Yes, and ... that ... that I'm quite ha ... well, fairly happy with the argument I can put for that one.
10	Tim:	[rising pitch] Alright.
11	Jonathan:	That's, that's, that's the happiest argument that I've got [laughs].

My laboured formulation [4] of Jonathan's progress indicates my embarrassment [6] that my recall is uncertain. It is a "deferential use of hesitation and bumbliness" (Brown and Levinson, 1987, p 187) which shows my reluctance to reveal to Jonathan that my memory of our last supervision has merged with that of similar conversations with other students. I try to redress the FTA with a request for acquittal [6]. Jonathan [9] stumbles over his words as he asserts that has a proof for the case n=0, and hedges his satisfaction ("quite/fairly happy") with his argument.

I urge Jonathan to state the theorem for $n=0$, and, after two or three attempts, he tells me that there is just one solution ($x=y=0$) when $p\equiv 3$ mod 4, and $2p$-1 solutions when $p\equiv 1$ mod 4 [NT4:12-27]. At this point (for I, the analyst of this conversation, have particular access to the intentions of one of the participants) I wonder whether we'll ever get around to the proof, and am anxious that Jonathan be aware of the significance of $p\equiv 1$ mod 4 relative to the theorem he has enunciated. In retrospect, I should have asked him to elaborate his argument first - once again, "the tape recorder trains the teacher". My intervention [28] is an imposition on him, since he has told me he has "an argument"; reluctance to perform the FTA is evident in the hesitation [28] which has no fewer than six false starts, as I search in vain for a way of not telling him the significance of the value of p modulo 4. The best I can manage is to avoid telling him which case is which!

28	Tim:	OK, OK. And, I mean, can I, I think, I just want to ask, does it hinge on the fact that in one case minus one is a quadratic residue and in the other case it isn't?
29	Jonathan:	[pause] Um ... well, yes [coughs] ... sort of. Um, I mean /it's, yes there's one/
30	Tim:	/[laughs] Would/ you like to rehearse the argument with me, or ...
31	Jonathan:	Well [coughs], yeah (yes), I'll come back to that bit about the quadratic residue bit. Um, but for where it's equal to one mod four ...
32	Tim:	Righ

The "well" that initiates Jonathan's response [29] seems to be a hedge on the maxim of Quality. After coughs and pauses, the best he can claim is "sort of". I soon realise the futility of this 'beating about the bush' and invite Jonathan [30] to tell me his argument, but still redressing face (his) by giving the option ("or") so that consent is not the only preferred response. He accepts the alternative, realises that this might disappoint me and challenge me in my role as supervisor. So although his answer [31] to my question [30] is (for the moment) "no", he presents it as "yes", appropriately marked by a hedge on Quality ("Well"). Thus, he asserts his right to present his argument in the way he chooses, but bears my prompt [28] in mind and eventually responds to it [39]

39	Jonathan:	Um, and then, if p is congruent to one [pause] we've then got ... p minus, p minus one is a quadratic residue of that.
40	Tim:	Indeed; or minus one.
41	Jonathan:	Whichever way. So that gives us, um, one squared and p minus one squared, so we've got another pair of solutions.
42	Tim:	Umm ... you don't quite mean that do you? I mean, you mean, what you've got, one squared and whatever gives you p minus one ...
43	Jonathan:	Oh, yes, so ..
44	Tim:	... when it's squared. Right, /OK/

Jonathan's account here is presented throughout in first person plural form rather than the more usual colloquial second person "you". The 'we's and 'us's run through his explanation [NT4:33-41]. This is, of course, the classical means of rhetorical distancing [Note 4.3], typically used by writers and speakers in formal mathematical discourse (Pimm, 1987, p 67). Jonathan is a sufficiently sophisticated (or encultured) mathematician to deploy it. At the same time, he may be wanting to include me, to take me along with his account. I recall that this was typical of his oral contributions in lectures, if I posed a question to the class. As one of the most able mathematicians in his year group, his accounts would be in terms of 'we', I sensed, in order to include the rest of the class in his insight.

As undergraduate mathematics students go (at least, the ones I have worked with over the last fifteen years), Jonathan is reasonably articulate when talking about mathematics. Yet in [41] he fails to say precisely what he intends to convey, and I am left to fill the gaps in the elided statement [39] ("mod 4" should follow the first "one") and to appreciate that the demonstrative pronoun "that" is co-referential with "*p*". The ambiguity in [41] is sufficiently serious that I feel that I have to clarify the meaning by correcting him. The consequent FTA [42] is redressed with the usual hesitation, hedging and indirectness ("you don't quite mean that, do you?").

Among the linguistic (as opposed to affective) barriers that stand between students' ideas and their articulation, two seem to be paramount. The technical language of the mathematics register seems at first to belong to the teacher; students must be enabled to inherit and appropriate it for themselves. One technique with which I have recently begun to experiment (in the teaching of Number Theory) to enable this appropriation is to make strips of paper with one 'technical term' on each. Examples include 'perfect square', 'quadratic residue', 'Euler's criterion', 'primitive root', 'discriminant'. From time to time (at suitable points in the course) the strips are randomly distributed to the class. Each student is given a few minutes to construct a sentence such as '2 is a quadratic residue mod 17' which includes the technical term on their strip of paper. They then speak their sentence to the class. Definitions (which could be found and quoted from notes) are not allowed, neither are meta-sentences such as "I never could understand quadratic residues"! The effectiveness of this technique is currently under review.

The second barrier, especially in discussion, is the syntax of 'proper' mathematical sentences, which rarely matches the cognitive production of elements of the sentence. Thus, in [41], Jonathan is conscious that 1 and p-1 are quadratic residues ('squares') modulo p when $p \equiv 1$ mod 4. Whereas 1 is equal to the square of itself and therefore serves for 'x', p-1 is not. The degree of detachment necessary, to hold the fragile mathematical idea in mind whilst the sentence is assembled with the correct syntax, is considerable,

Soon, it becomes apparent that Jonathan's argument, with which he is only "fairly happy" [9], is incomplete.

47	Jonathan:	And, then there's this pairing thing ..
48	Tim:	Yeah?
49	Jonathan:	Which ... that's the bit I can't, I'm not ... able to explain. I can't, I'm not, I can't say why they pair off, like that. Um, but then(?) we've got, um, p minus one over two pairs [number of q.r's mod p] [inaudible]
50	Tim:	Oh, p minus one over two squares.
51	Jonathan:	Yes. And so, so you get [long pause] yes, sorry, yes that's it. And they add up to give p each time, these two .. these pairs of squares ..
52	Tim:	Yes
53	Jonathan:	So you've got p there, nought.
54	Tim:	[pause] Um, [hesitant] that's an absolutely fine ... um, I mean, let's think, we're talking about when p is congruent with one mod four here, aren't we?

Jonathan identifies the gap in his argument, which is to show that, when p=1 mod 4, the quadratic residues can be always be paired to give sum zero. In [54] I 'formulate' the conversation, in the sense of Garfinkel and Sacks (1970) described in Chapter 5 - "we're talking about ..." - and try to insist in the formulation that these are his ideas ("you're saying"), not mine:

58	Tim:	So you're saying .. um, um .. I'm trying to think of something that isn't *[equal to13]* ... well, no, let's have something fairly straightforward. If you do two squared you get four, yeah?
59	Jonathan:	Right.
60	Tim:	And you're saying that in fact, um, thirteen minus four, or something congruent to that, minus four, mod thirteen, is always - in fact it's nine ...
61	Jonathan:	Yes.
62	Tim:	... three squared - is always there. So you're saying, in that case, there always happen to be pairs that add to ...
63	Jonathan:	Yes.

I then acknowledge the gap in the argument, redressing the FTA with a positively-hedged compliment "that's absolutely right" [64, c.f. 54]. Jonathan's response indicates, for the first time, his discomfort. The gap in the proof is now exposed, and he can offer no resolution to the problem.

64	Tim:	OK. I mean, can you take it any further than there. I mean you're absolutely right. How can you take it any further than "there always happens to be"?
65	Jonathan:	No, I'm stumbling on this, but this is, this is the the bit that, it's sort of an assumption I have to make [exhaled laugh] to go through this, and I ..
66	Tim:	OK
67	Jonathan:	... I can't ...
68	Tim:	OK
69	Jonathan:	I can't ... and I know ... or, I don't know ... from looking at the ones that are congruent to three, mod four ..
70	Tim:	Yes
71	Jonathan:	... that there's not a constant adding up, for the pairs, so I can see that the two ..
72	Tim:	Right.
73	Jonathan:	... they really do separate, but I can't explain, why they separate.

The exhaled laugh [65], like the cough, indicates Jonathan's uneasiness. He must make an assumption to "go through this". "This" is deictic, its referent an argument, hopefully not an ordeal. I adopt the minimal response strategy [66-72] (see Case 3: Judith) in the hope that he might be holding something back. But in this case there is no insightful outpouring from Jonathan. Rather, he insists that he "can't explain" [73].

My next turn [74] somewhat apologetically initiates Episode 2 and my explanation.

74 Tim: OK. Well, I'd like to take you a bit further down that road, because I think you'll be quite pleased when you see it. OK?

75 Jonathan: Right.

76 Tim: I'm just wondering whether to talk about thirteen, or something that's less obvious. You know, 'cos (because) [laughs]...

77 Jonathan: Oh dear [laughs], is thirteen obvious! [laughs]

78 Tim: No, no, no, I mean, um, an argument can be more forceful when you can't just - other than the numerical calculations - say "Well, obviously". Yes?

79 Jonathan: Yes.

In [76, 78] I formulate the account which will follow. Since Jonathan was preparing to be a teacher, I was explicit about the pedagogical strategy that I was about to choose, i.e. the use of a generic example [78].

We are less than a third of the way through the transcript.

I find, looking back at the text, that I was at pains to affirm Jonathan's achievements - which are significant, especially towards the end of the transcript. At the same time I have to correct errors and suggest approaches to proofs - not least because this was Jonathan's last supervision with me, and he was about to write up the project report over the Easter vacation. The elements of a polite contest can be seen in Episode 3 (reproduced below [136-156]) - compliments given [144, 150, 152] and accepted [153], Jonathan maintaining his approach, refusing sometimes to take the route I am suggesting [137], asserting his achievements [145, 147].

136 Tim: OK, OK. [pause] So which then brings us, I guess, back to x squared plus y squared equals one, does it?

137 Jonathan: Uh, yes. Yes. That's a very intriguing way of ... well, actually no, I went on to do something else first.

138 Tim: Ah?

139 Jonathan: Of x squared plus y squared equals n mod p, but n not being zero and

having a particular value.

140 Tim: OK.

141 Jonathan: Um, having found out how many solutions there were for that ...

142 Tim: 'cos (because) originally you took it to be anything other than zero ...

143 Jonathan: Yes.

144 Tim: Which actually is not without interest, if I may say so.

145 Jonathan: No, I think I did go on to show, um ...

146 Tim: I mean, now you know how many solutions there are for zero, you can say precisely how many there are for not-zero [laughs]

147 Jonathan: Yes, that's basically what I'd done! [laughs]

148 Tim: OK.

149 Jonathan: p squared and taken away how many other bits there are ...

150 Tim: Excellent.

151 Jonathan: ... so I've done that.

152 Tim: Yes, that wraps up quite nicely too.

153 Jonathan: Yes, that was very satisfying, actually.

154 Tim: Yes. I think it's just a nice coincidence that you've two p minus one solutions, and when you subtract that from p squared, you get a perfect square ...

155 Jonathan: Oh, yes ...

156 Tim: p squared minus two p plus one is a, an algebraic square.

The encounter certainly had its lighter moments, such as when Jonathan pays me a compliment (in that he recognises my own mathematical sense of curiosity) which I gratefully accept, and he immediately eases my embarrassment (young students hardly ever praise their lecturers; doubtless they lack the confidence to do so) by teasing me for my untidiness:

203 Tim: Can I tell you that I don't know the answer to this [both laugh]. I mean simply because I've not allowed myself to think about it ...

204 Jonathan: Yes, very restrained of you. [both laugh]. As soon as I'm out of that door you'll be going ... [inaudible beneath Tim's laughter]

205 Tim: Nice of you to suggest it. Um ...

206 Jonathan: Anything to put off sorting through that pile of papers! [gestures to Tim's desk]

207 Tim: [laughs] Dead right!

In the fifth and final Episode of the conversation Jonathan surprised and delighted me with a proof which involves a neat combinatorial argument. I begin the episode by suggesting [209] that he will find the proof too difficult. Jonathan's reply is dispreferred [210] and the FTA is marked and redressed by the three false starts. I realise that I have underrated him [211].

209 Tim: You know the other thing that, um, you haven't proved - but in a way I don't feel to desperate about it because there's quite a lot around here for you to write up - is ... is why there are ... one more or one less than p solutions to x squared plus y squared equals one, in every case.

210 Jonathan: Ah well, ah, that's, I'm coming on to that bit ...

211 Tim: Ah, right! Sorry ..

212 Jonathan: I had to backtrack to get to that.

213 Tim: Oh, right. OK, OK.

His proof [NT4:215-234] is perfectly sound. Jonathan sketches it sufficiently for me to know how it is structured, and that it achieves the desired conclusion. [Note 7.4] It rests in part on one of Jonathan's earlier theorems, the one that I had rather loftily consented [144] to be "not without interest".

Nonetheless Jonathan is quite diffident about his proof, and Shields himself [216] with some language reminiscent of the visiting scholar (Case 6) - "arm waving", "sort of proof" c.f. 'hand waving', 'almost proof':

216 Jonathan: Um ... I'm can then get back to some serious arm waving here and ... and go back to my sort of proof of why there are x ... there are p plus one or p minus one solutions.

This is *his* proof, and its generality here is marked with "you" [218, 220], whereas he uses the expository "we" [224, 226] to refer back to results agreed earlier in the conversation.

218 Jonathan: And, basically, um, you say how many ... you take your mod, number ...

219 Tim: Right ..

220 Jonathan: And you work out how many possible pairs you can come up with ...

221 Tim: [hushed] Right ...

222 Jonathan: And ... whatever that was, p squared ...

223 Tim: OK.

224 Jonathan: We already know how many... solutions there are - for p congruent to one or congruent to three mod four - how many solutions there are for ... x squared plus y squared is congruent to zero [pause]

225 Tim: [hushed] Yeah ...

226 Jonathan: So we can get rid of those for starters. And then we know that all the solutions that are left are divided up evenly between each of the other numbers ...

My response was little short of ecstatic:

227 Tim: Ohh, that's very nice. [Jonathan laughs]. Oh, well done!

233 Tim: Oh, well done, well done. Yes ... [laughs] [...]

234 Jonathan: I didn't know if that was the way you were thinking of ...

235 Tim: That's very nice indeed.

I urge him to write it up without delay; Jonathan responds that he has done so already.

241 Tim: Right. Well, I would suggest you rush away and write all this down.

242 Jonathan: [laughs] Well, I wrote it all down yesterday, that ... that particular bit ..

Both turns [241, 242] are affronts to 'face' (my imposition, Jonathan's refusal), and both are redressed in the same way, with "Well". There is some new work here, he acknowledges:

244 Jonathan: ... so it's just the, um, it's the pairing up of the quadratics when it's congruent and ... that tidying up of the, er, x squared

And so the supervision concluded:

247 Tim: Rush away and write it up.

248 Jonathan: Yes, before I forget it!

249 Tim: [laughs] OK. Well that was a good way to start the week.

Jonathan did indeed write it up, and submitted his report, neatly word-processed, on time at the beginning of the Easter (Summer) Term. I read it and awarded 16 marks out of 20 (14+ is First Class). My co-examiner in the Faculty of Mathematics, on reading the script, remarked how much Jonathan must have enjoyed this project, and I happily accepted his proposal to upgrade the mark to 18.

One month into that Easter Term, on 16th May 1995, Jonathan took his own life, alone in his house in Cambridge. He was a complex person, one who was more comfortable giving than receiving. Yet his innermost feelings he reserved for himself. I imagine that, for Jonathan, life posed a number of threats and challenges, and that, in the end, he did not believe that he could face all of them.

SUMMARY

In this chapter I have examined a number of mathematical teaching and learning situations. I had no part in or control over any but the last of these pedagogic episodes, but have used them to validate the claims I arrived at and set out in earlier chapters. These claims concern the many ways in which indirect and vague language are used to support interaction in the mathematics classroom, and serve the interactional and transactional intentions of teachers and students. I have shown that the linguistic phenomena which I had identified in my own research-oriented, contingent interviews, are also present in these pedagogic encounters. I have shown how these phenomena assist the definition and interpretation of the propositional attitude of speakers (students and teachers) and the dynamics of respect and politeness in a conjecturing atmosphere. In particular, I have identified in these eight episodes:

- the deictic function of pronouns 'it' and 'you' for generalisation
- the subtle use of indirect speech acts for the redress of threats to 'face'
- the occurrence of hedges and modal forms, mainly implicating uncertainty
- reification of the Zone of Conjectural Neutrality in a conjecturing atmosphere

More generally, I have demonstrated the possibility of insight into mathematical interaction by means of pragmatic analysis of discourse. In the interests of careful interpretation - so that it might be something more than "a matter of guesswork" - I have applied the full range of the pragmatic analytical methods which I reviewed in Chapter 2. Classroom talk about mathematics is both transactional and interactional.

Interpretation of the interactional component of such talk requires attention to a wide range of human sensibilities and pragmatic goals.

In the final chapter I shall conclude with some remarks concerning the value of this study for my own understanding, together with some proposals for application.

CHAPTER 8: SUMMARY AND REVIEW

> So what did we learn? (Margaret Thatcher, 1983).
>
> Wisdom is the principal thing; therefore get wisdom: and with all thy getting get understanding. (Proverbs 4:7)

In this thesis, I have taken the position that, when people talk about mathematics, they use language as a means of satisfying a number of communicative 'wants'. My data is drawn almost entirely from expert-novice conversations, and my conclusions apply, in the first instance at least, to such discourse contexts. Broadly speaking, communicative wants in mathematics talk are of two kinds.

The first kind is cooperative and cognitive, stemming from a desire to share mathematical ideas - to give and receive insights, knowledge and understanding. It is associated with transactional functions of language. This giving and receiving is not one-way information traffic, flowing from teacher to pupil. The teacher needs to know what the pupil knows, what kind of knowledge s/he has constructed. My transcripts amply demonstrate the willingness of pupils to supply such information, and on occasion the pleasure they derive from doing so.

The second kind is of a more general, social character, to do with establishing and sustaining social relationships. It is associated with interactional functions of language. In the case of mathematics talk, language must frequently serve the cause of respect and defence of 'face'. This is to be expected as a consequence of the asymmetry of the power relation in the expert-novice conversation, since the novice anticipates that the expert will evaluate her/his assertions. As I have observed in Chapter 5, the line between truth and error is perceived to be particularly sharp in mathematics in comparison with other subjects at school and college. But this is not only a 'problem' for novices. Experts, too, recognise the boundaries - albeit fuzzy boundaries - of their expertise, and use language to convey uncertainty to their audience when they make judgements and predictions on matters on or beyond the boundary.

OUTCOMES

My aim in this work has been to highlight and analyse some of the ways that participants in mathematics talk use language to achieve their communicative and affective purposes. In this thesis, I have exposed and analysed the communicative *competence* [Note 8.1] of pupils in the use of mathematical meta-language, especially the goals they achieve through the use of various kinds of vague language. In this research, my attention has focused on the following topics and issues:

- the use of pronouns by novice speakers of mathematics;
- the function of hedges in the communication of propositional attitude;
- the construct which I have called The Zone of Conjectural Neutrality;
- the development of the language of modality over the years 4 to 11;
- the use of vague and indirect language by teachers in interaction with pupils;
- pragmatic interpretation of transcripts of mathematics talk.

In this chapter, I shall recapitulate some of these matters, and review the potential for more general classroom application. I conclude by identifying areas for further research.

PRONOUNS

In the study of reference vagueness in the use of pronouns in mathematics talk (Chapter 4), the pupils' deictic intention is principally cognitive - pointing to concepts and generalisations - rather than affective. The significance of the teacher's 'we' has previously been considered by Mühlhäusler and Harré (1990), who propose that its function is commonly manipulative (spurious solidarity); also by Pimm (1987), who argues that its function is authoritarian (appeal to unnamed expert support). I have shown that he pupil's 'it' and 'you' are equally vague in terms of intended referent, but are not related to the teacher-pupil power imbalance. I have argued for a strong association between their use and reference to concepts and generalisations. Given the importance of these referents for mathematics teaching, the significance of this connection can hardly be over-stated.

The use of 'it' as a conceptual deictic enables the pupil to say what s/he could not say otherwise, to draw attention to mathematical entities whose name s/he does not know. In terms of the 'two language levels' analysis presented in Chapter 3, this pronoun (typically as object of the verb 'to do') is added to the object language as a vague variable. The notion of 'focus' as locus of attention is important here, given the claim of Moxey and Sanford (1993, p. 58) that "pronouns are good for referring to things in focus [...] ease or acceptability of pronominal reference can be used as an index and a probe for the state of focus". In the first instance, the teacher who is sensitive to the pronoun/focus connection can be made aware of the presence of a cognitive focus such as a generalisation - recall Susie's "times can do it can't it, and add, and take ... no, takeaways can't do it". This teacher-awareness opens up the possibility of further investigation of that focus through appropriate, contingent questioning.

This questioning could be of two kinds, which might be called conspiracy and confrontation. The conspiratorial approach is for the teacher to take up and use 'it' in the discourse as though her/his 'it' were intended to be co-referential with the pupil's. For example, "Why can't takeaways do it, then?" would enable the confirmation or formation of hypotheses about the referent on the basis of further information about 'it' - rather like a game of 'twenty questions', the only question not permitted is the name of the mystery object. Confrontation, on the other hand, amounts to an 'on record' request for the object to be revealed. For example, "Wait a minute, what is this 'it' you're talking about?". Clearly the choice of the conspiratorial or the confrontational approach must depend on a range of contextual and inter-personal factors, and there is scope for some research here. My default choice would always be conspiracy, but that reflects my preference for the avoidance of on-record FTAs in the absence of detailed knowledge of the pupil/student.

In Chapter 4, I discussed how 'you' serves as a pointer to a generalised procedure or relationship. The subtle shift from 'I' to 'you' to mark a tendency towards speaker detachment is an important cognitive indicator. Oscillation between first and second person pronouns indicates a switch between action and knowledge, possibly with regard to different processes or generalisations.

The 'I' - 'you' contrast could be related to the process-object distinction (Sfard, 1991); the detachment associated with the conception of a mathematical notion as an object independent of the action of the speaker is marked by 'you'. [Note 8.2] The pedagogic significance of the pupil/student's use of 'I' or 'you' might be in the recognition of a pre- or post-objective cognitive state with regard to the mathematical notion being discussed. There is far greater ambiguity in the teacher's use of 'you' in that s/he may either be addressing the pupil (meta-language) or referring to a mathematical notion (object language). In practice the referent is pragmatically determined, and/or determined by tense and mood. (I shall not develop that thought here, but compare "How do you find the area?" with "How did you find the area?" and "How would you find the area?"). My data suggests that pupils hardly ever use 'you' to address the teacher in *mathematical* conversations.

MODALITY, HEDGES AND INDIRECT SPEECH ACTS

Hedges encode vagueness in mathematics talk, related to imprecision or uncertainty of speaker commitment. Whereas Shields clearly belong to the meta-language of mathematics talk, Approximators affect to supplement the object language and confuse its truth-conditional semantics: this perception is the motivation of Lakoff's classic 1973 paper on the logic of fuzzy concepts. In the end, a pragmatic analysis of hedges is more fruitful for the purposes of mathematics education.

Having identified hedges and epistemic modal forms as a feature of mathematics talk in a conjecturing atmosphere, much of the discussion of their use in Chapter 5 focused on the pragmatic goals achieved by speakers, as analysed by Channell. These purposes include covering for lack of specific information and expressing politeness. In these pragmatic terms, the following important differences between pupil and 'teacher' were identified.

Essentially, pupils use epistemic hedges to shield themselves from accusation of error; the most subtle form of this is the epistemic use of Approximators, so as to render a statement "almost unfalsifiable" by trivialising its semantics (Sadock, 1977, p. 437). The development of this aspect of communicative competence (hedging), vital for protection of 'face' and the communication of propositional attitude in the mathematics classroom, is traced and interpreted in Chapter 6.

Teachers, on the other hand, may use hedges as a tool of pedagogic strategy, weakening the force of an assertion (her/his own or that of a pupil) in order to sustain the engagement of (other) pupils, emphasising their responsibility for the determination of validity. At the same time, teachers use epistemic hedges and modal forms to perform indirect speech acts in interaction with pupils, particularly in order to present a request (for information) as a question (default illocutionary force of the interrogative form) e.g. "What do you think this shape is called?", or "Can you tell me the mean of these four numbers?". The purpose of these indirect speech acts is to alleviate the illocutionary force of an act that could lead to 'loss of face' if the pupil cannot supply the 'answer'.

The application of these findings to the teaching of mathematics is probably easier to state than it is to put into practice. The possibility of active construction of knowledge from reflection on experience is at the heart of a constructivist view of learning. Such a view puts an onus on the teacher to try to understand the form, content and robustness of that knowledge, as an observer of and participant in pupils' mathematical activity -

an "accultured" participant, moreover, who "can legitimise certain aspects of their mathematical activity and sanction others". (Cobb, 1992, p. 102) For this reason, amongst others, it is desirable that pupils regularly articulate their constructed beliefs, or construct them through articulation, in the hearing of their teacher. Such a self-constructed belief may be fragile; in particular, any inductive conjecture would be expected to be. The burden of the affective baggage associated with mathematics in school then necessitates that the pupil articulate the belief whilst distancing her/himself from full commitment to it. That is to say, they must convey their propositional attitude to the substance of their assertion. The rich variety, in some cases the subtlety, of hedges and modal forms deployed by pupils for this purpose is evidence of this affect-oriented dimension of pupils' communicative competence. These Markers are linguistic pointers to uncertainty and attendant cognitive vulnerability. The teacher's subtle task at such moments is to facilitate the de-personalisation of the assertion, as a preliminary to 'legitimising' or 'sanctioning', by ensuring that it be located in the Zone of Conjectural Neutrality. This will be discussed further following the next section.

My comment about the relative ease of stating such a policy as opposed to implementing it is based, in part, on my own experience over the last two years. These proposals are no quick fix for teachers, and in any case presuppose a quasi-empirical approach to mathematics learning. As Piaget said with reference to clinical interviewing, sensitivity to epistemic Markers in mathematics talk also "can only be learned by long practice". Despite my awareness of the form of such Markers, the difficulty (for me, when teaching) is in *attending* to the linguistic details of a 'lesson' whilst being fully engaged with the mathematics. This difficulty is clearly related to one of Hewitt's pedagogic aphorisms: "The amount of attention available to us is finite and limited" (1994, p. 69). In particular, I suggest that it is difficult, if not impossible, without "long practice", to attend to two *demanding* things simultaneously. If the level of demand of one task can be reduced by 'automation' (as in 'being on auto-pilot') as a result of familiarity, thereby markedly reducing its attention requirement, then simultaneous *performance* may be possible (e.g. taking in the news on the car radio whilst driving on an open road). Perhaps some aspects of mathematics teaching can be automated in this way, but I don't know; my preference for some time has been for 'contingent lessons' in order to *avoid* the tedium of automation.

It would, I imagine, be equally tedious (in the classroom) to be continually and consciously preoccupied with certain syntactic features of speech ("that was a subordinate clause"), even those thought to have particular pragmatic significance. Nevertheless, I have made some progress in achieving the automation of sensitivity to

Marked language. Within the last year, in a lecture to first year students in a course on Mathematical Processes, I 'invited' them to evaluate a^2+b and b^2+a when $a+b=1$. An inductively-based conjecture was proposed and proved by the students. Various "what if?" variants followed e.g. if $a+b=10$, is there another "interesting" function of a, b, symmetrical on the given subset? What if a, b are complex numbers and $a+b=1$? The algebraic requirements of the proof indicated that we could go further. What if a and b are 2x2 matrices? There was a few moments' pause, then Verity spoke: "Would they have to add up to the identity or something?". In that moment I recognised the question as an indirect speech act, asserting a tentative belief, further shielded by the vague completer "or something" working as an epistemic Adaptor. I was very aware of the significance of what she had said, yet my attention had been on the mathematics until that moment. It was the first time that I suspected that attention to a restricted range of lexical pointers (pronouns and hedges, for example) could be automated "by long practice". If teachers can be attuned to such pointers to propositional attitude, yet more work is needed on the guidance of appropriate teacher behaviours when a the use of these pointers is noticed in the mathematics classroom.

THE ZONE OF CONJECTURAL NEUTRALITY

In introducing the construct of the ZCN in Chapter 5, I described it as a space between what we believe and what we are willing to assert. This is, of course, a somewhat metaphysical construct, and deserves a name only if it captures something of didactic significance.

Now, the issue central to the notion of ZCN is summarised in the question "Where are pupils' conjectures located? Who is responsible for them?" The default position must be that a conjecture belongs to the one who utters it. If the conjecture is asserted with conviction (better still, if it is subsequently validated as true), then this is not an affective problem. But if a conjecture is offered tentatively, then it is better that it be located somewhere neutral before it is tested, in order that there be some real prospect of Dawson's promise (1991, p. 197) of "testing on a cognitive rather than an affective level". I emphasise again, that this is in defiance of the cultural norm that the pupil is judged to be 'right' or 'wrong' rather than the 'answer' 'true' or 'false'; that it is s/he who is on trial, not her/his beliefs.

This is at the heart of pupils' communicative competence in the use of Marked language in the assertion of conjectures. The forms of linguistic shielding which I have discussed have the effect of reifying the ZCN and locating the conjecture in it, thus distancing the speaker from the assertion that he or she makes. A Plausibility Shield such as 'I think', 'maybe', 'perhaps' *does* this (being a hedged performative, a speech act) in a very direct way, because the marker of propositional attitude lies outside the statement that follows it. Epistemic Approximators (such as 'about' in "there are about fifteen ways") are more subtle: they do not require the speaker to disown her/his conjecture, but they do make it almost unfalsifiable. Whilst subtle, this is less than helpful since a consequence of its vagueness is that, strictly speaking, it can neither be validated nor modified. The conventional force, however, is clearly to present the conjecture as fallible, possibly in need of modification.

The teacher who recognises the epistemic force of a Marked conjecture has the option of assisting its placement in the ZCN. One way to do this might be to write it on the 'blackboard' and say something like "OK, let's take a look at this conjecture", possibly without reference to the one who proposed it or constant application to him/her for arbitration or interpretation. Another way is to form small discussion groups which then tend to assume some corporate ownership for the conjecture and their findings about it when reporting back to the class. I sometimes 'return' a conjecture, or an agreed modification of it, from the ZCN back to its originator when the "severe testing" is over; I do this, for example, by marking its changed status with reference to the conjecture as 'theorem' (sometimes 'lemma') and naming it Yuko's Theorem or Tom's Theorem. If Fermat and Langrange merit such attribution, then so do Yuko and Tom.

The term 'conjecture' may suggest the cognitive outcome of an extended investigation, but it could be simply the answer to a teacher's question. "Is 91 a prime number?". "How many non-isomorphic groups are there of order 8?". By default, the one who answers the question 'owns' the answer and is subsequently right or wrong. One way of trying to bring the answer into the ZCN before it is spoken is to pose the question as an indirect speech act. "Can you tell me if 91 is a prime number". As I have observed (Chapter 3) the illocutionary force of the indirect act is achieved by the modal auxiliary, which questions one of the felicity conditions (ability to comply) of the direct request for information. Another, rather different technique, is to pose questions as statements (with the tacit or explicit "Discuss"). Thus, "91 is not a prime number". Or by attribution: "My neighbour says that 91 is a prime number". The conjecture then goes straight into the ZCN; at the very worst, only the teacher (or his neighbour) are 'wrong' if the statement turns out to be false. But this technique has limitations, and cannot help with extended enquiries "in which a conjecture is created, tested and proved, or refuted and modified" (Dawson, *ibid.*)

In a conjecturing atmosphere, a pupil may articulate a conjecture without necessarily being committed to its truth. Both the pupil and the teacher may influence the relocation of the conjecture from the pupil to the ZCN. The conjecture is then tested, modified or rejected in the ZCN. In such a cognitive and affective *milieu*, it is the proposition that is on trial, not the person. The ultimate goal, for the fallibilistically committed teacher, would be for the class to understand that this is the case.

VALIDATION AND CLASSROOM APPLICATION

In our first three evening meetings, the eight regular members of the Informal Research Group were exposed to the linguistic framework and most of the findings reported in Chapters 5 and 6, set in the context of contingent questioning and the use of transcripts to interpret classroom interactions. The next two meetings were given to feedback and study of the transcripts that six members of the group had made of their own tape recordings. The discussion included general consideration of the benefits (if any) of attention to vague aspects of mathematics talk in the classroom. Four of these teachers (Judith, Rachel, Sue and Hazel) responded to my request for written reflections. This was a formative experience for all of us, and was insufficiently focused to enable adequate evaluation of my proposals for application of this research.

Five of the transcripts considered in Chapter 6 were donated by members of the IRG. In each case, I sent the donor a copy of my analysis of their transcript (i.e. the relevant section of Chapter 6) inviting them to correct and/or comment on it, and asking their permission to include it. Four of the five replied. Apart from one or two minor corrections (such as the age-range of Judith's school), these four were happy that my interpretation of their transcript concurred with their recollection of their interaction with the pupil(s). Hazel's comment was typical:

> In reference to the draft chapter that you sent me, I agree with all that was written and am happy for all of it to be included. I particularly agree with two points that you made in the second page [*etc.*]

I also took it as positive that six teachers, representing four phases of schooling from 4 to 18, gave up three evenings and two Saturday mornings to be involved in the group (not to mention recording and transcribing interviews with children, some of them lengthy texts). Their incentive appeared to be intrinsic interest and/or a sense that this might have potential for the improvement of mathematics teaching and learning in their classrooms.

In terms of the specific focus of the group on vague language, it appeared that the phenomenon of hedges was one that they could readily identify. It was also the only one. This may relate to my earlier comment about the difficulty of attending to language in the classroom as distinct to attending to the business of teaching mathematics. It may equally well reflect the thrust of my exposition to the group, which did not attempt to embed hedging within speech act theory and presented it more in phenomenological terms. Nonetheless, informal feedback suggested that these teachers had been sensitised - or believed that they had been - to the use of hedges by children as an indicator of propositional attitude, principally of uncertainty. It was not always clear what they then did with that knowledge, although Hazel wrote (personal communication) that:

> I am now more aware of the effect of using vague language in the classroom so I can use it in a positive way. [...] My knowledge of hedges has helped me to spot that some some statements made by children are less certain than they appear. [...] I can then respond to them at an appropriate level.

Sue had developed the task which was the basis of her transcribed talk (five people, how many more are needed to make ten?) into one which began with (say) eight people, then seven, six, and so on, so as to offer some potential for the child to predict and generalise. She had also begun to work on the way she presented questions (in effect, as indirect speech acts) to the four- and five-year-olds in her class, searching for ways that would convey her attitude that their answers were modifiable conjectures. Sue wrote (personal communication) that:

> thinking about this project [IRG] has influenced the way I have spoken to the children - I have only tried a small group with [the] second exercise but when I do I feel I shall find a few more "I think"s and "shall I try"s [...] when maths becomes sums which are right or wrong we stifle some children's embryonic sense of pattern in number and their enthusiasm for investigation.

Perhaps Sue has come closest (not only in this quotation) to articulating an understanding of the significance and possible application of the work of the IRG with regard to the communication of propositional attitude in a conjecturing atmosphere. Not surprisingly, the focus on vague language was often obscured by other, equally important insights into teaching and learning in their classrooms which the teachers gained from recording and transcribing their interactions with pupils. Judith wrote that "This [taping and transcribing] was an insight for me into the level of my own pupils' understanding, and into some of their idiosyncrasies of thinking". One specific instance, mentioned in Chapter 7, was Sue's realisation that her intended, restricted meaning of 'more' was widely misunderstood by young children.

Rachel listed a number of general conclusions and questions which arose from her study of her transcripts, including the following:

- The conversation is very dependent on the rapport between child and interviewer.
- Does the nature of the conversation change after a number of interviews?
- Is there a danger of the interviewer becoming familiar with the child's way of thinking and then beginning to interpret for them?
- Traditionally maths has been an exception to the use of children's talk as a vehicle for learning mathematics. Maths has a symbolism which seems formidable. Perhaps this has caused some teachers to base their language work on the transmission and use of symbols and on learning the formal, spoken vocabulary.
- Students often discuss things in lessons with me and each other and then say "But how do I write that down?"
- Maths is a precise subject - is uncertainty valid?
- I am impatient that they should get to the correct answer.
- I wonder what the students would gain from listening to the tape?

(Williams, 1995)

A comment made by Judith was illuminating. She remarked, in discussion, that the work on vague language had been "really interesting" but that she did not think that it was useful, she couldn't see how she could use it. Hazel and Rachel (three-year veterans of teaching) responded - with all kinds of redress of FTA - that they did feel that it was useful to them as teachers, and that Judith's perception may be due to her inexperience as a teacher. They recalled that they had been preoccupied with preparation and classroom management in their first year or so of teaching, and only recently had been in a position (having automated at least some aspects of their teaching?) to think about fine-tuning their interaction with pupils. Judith listened to this suggestion and agreed that it was a possible explanation. Sue and Ann (the real veterans) wisely withheld comment.

At that time I had not analysed Judith's transcript [IRG5]. What is interesting, in the light of Judith's honest perception, is the richness of her conversation with Allan in terms of mathematical process (see Chapter 7: Case 3) and her skill in creating and sustaining a conjecturing atmosphere. It hardly matters, for her, that she is not (yet?) able to 'use' language in an analytical way to achieve or develop what she already does intuitively. Some time later, Judith attributed her good 'instincts' to the influence of a particular Cambridge University tutor on her PGCE course:

> I myself was trained to look beyond language to meaning - whether I was creating an environment where students were free to make decisions and predictions, to make mistakes and correct them ... (personal communication).

In retrospect, more systematic data collection from the Informal Research Group would have been useful, not to say preferable. This is particularly the case in the area of classroom application of my basic research on linguistic pointers (especially pointers to uncertainty). My limited success in obtaining evaluation data was not for lack of trying. In practice, I was totally dependent on the goodwill of the members of the group, since their only incentive to participate at all was personal and professional interest. Despite the fact that I set out a 'contract' for participation in the group at the beginning (attendance, supplying a transcript, contributing to a publication), I was in no position to enforce it. What I could, and should, have done was to arrange one-to-one interviews with the six teachers who contributed transcripts of their talk with pupils. Nevertheless, the IRG was valuable in its affirmation of the relevance, to their classrooms, of my research on aspects of vague language.

INTERPRETATION OF TRANSCRIPTS OF MATHEMATICS TALK

Much of the data in this thesis have been transcripts of people talking about mathematics in pedagogic or quasi-pedagogic situations. My interpretivist position denies the possibility of 'knowing', in any pure and absolute way, what a given utterance 'means'. The interpretation of the meanings and motives of others entails incorporating the evidence of the text into one's view of the world, the actors in the interaction, and one's knowledge of their situation. It is a constructive act of meaning-making for the analyst, whose 'reading' of a particular utterance must be made to fit, to be consistent with, the way that they construe the utterance in its context.

Moreover, the analysis of mathematical interaction must be more than the interpretation of individual utterances; it must account for how the discourse is sustained as a social event. The participants in the conversation must make such interpretations of mathematical and social meaning in the moment, within the conversation itself, to make an interactive contribution to it. The task of the analyst, in this fullest sense, is to give, with all possible skill and integrity, an account of the record of the conversation, transformed and preserved in the transcript. Such an account is a 'story' which need not claim to be true, but can endeavour to contain some *truth* about what it might be like to talk to someone - a teacher, a child, a student, a friend - about mathematics. How it feels to ask a question, or to be asked for an answer; how it is possible to say what you know, and how it feels when you do; what it's like when you know but don't know how to say it.

Perhaps, as Leech suggests (1983, pp. 30-31), interpreting an utterance *is* ultimately a matter of guesswork, but that does not mean that one guess is as good as another. I have shown that the pragmatic approach to discourse analysis offers a way of setting about this business of interpretive guesswork in a responsible way.

FURTHER RESEARCH

Some questions which have emerged from this research indicate the need for further research which would occupy a continuum from the basic to the applied.

My study brought to light some gaps in corpus-based research on child language. The wealth of research in language development tends to focus on the first three or four years of life, for the good reason that development is most dramatic in that period. Another reason is that children are most available for researchers to observe (not uncommonly their own children) before they begin school. This bias is also reflected in CHILDES, the largest electronic text corpus of child discourse available (MacWhinney, 1991). This poses some problems for comparing the mathematics talk of schoolchildren with home talk or any other kind of talk. CHILDES does, however, include the Fawcett corpus [Note 4.5] of the spoken language of 92 children aged between 6 and 12. A project comparing Marked language in this corpus with that in my own would be of considerable value and interest.

Teachers' use of questions as indirect requests is of interest, not least because the use of the device is (I imagine) almost invariably unconscious, and because it varies in the practice of any individual teacher and in extent from person to person. This is evident in the IRG transcripts. There is also variation in the form of the indirectness itself. Study of this as a phenomenon (a linguistic regularity) and as a pragmatic device (related to context and goal) would be illuminating.

Rachel's questions included the possibility of referring the evidence ("the tape") back to her students. A similar idea had occurred to me: what might the children and students whom I recorded say, in retrospect, about instances of vague language that I brought to their attention? It would be naive to suppose that they would then provide me with a reliable account of their goals; I anticipate that answers would contain a fair amount of hedging! Moreover, 'romancing' is a tendency and a temptation in the face of such introspective recollection. But it would have some value for triangulation purposes, and a pilot study would certainly be worth trying.

Are there linguistic pointers to ideas and attitudes in mathematics talk in addition to those identified in this study? There is scope for a fresh reading of Stubbs' programme (1986) and further scrutiny of my corpus with this question in mind. Intonation is one obvious suggestion; this would require a return to the tapes and a more sophisticated coding of the transcripts.

As regards further working with teachers on classroom application, some philosophical commitment (albeit tacit) would, I believe, need to underpin working on classroom behaviour. A necessary preliminary to directing attention to linguistic matters would be to work, over a sustained period of time, on the teaching styles associated with quasi-empiricism. Working with other teachers, I would want to look at some of the the writing of Polya and Lakatos, Dawson and Mason. I believe that we should need to grapple with the ZCN. That we would benefit from reading the work of Paley on the refinement of practice through the study of audiotape, videotape and transcripts. The IRG provided sufficient evidence for me to begin to believe that such a programme could work, at least with some teachers - especially given some incentive for those teachers, principally the availability of time.

SUMMARY

In this concluding chapter, I have summarised the outcomes of my research in relation to the aims that I set out at the beginning, and indicated some areas for further research. The main findings, reported in earlier chapters and reviewed above, can be summarised as follows:

1 Novice speakers of mathematics are able to make skilful use of the pronouns 'it' and 'you' to point to mathematical concepts and generalisations, and to indicate detached generalisation by a subtle shift from first person to second person pronouns.The details of the associated study are reported in Chapter 4.

2 Hedges (Chapter 5) play an important part in the communication of propositional attitude, and this is of vital importance in the formation and articulation of predictions and generalisations.

3 In a conjecturing atmosphere, associated with fallibilistic teaching and learning, epistemic hedges implicate uncertainty. The Zone of Conjectural Neutrality is a pedagogical concept, capturing the idea of an idealised space in which tentative conjectures might ideally be tested.

4 The language of modality, and of hedges in particular, develops in a more-or-less consistent way over the years 4 to 11. Whilst some children develop the root sense of approximators, for most children such hedges are deployed in the institutional (school) setting to protect against accusation of error.

5 These interactional dimensions of language are present across a wide age-range of pupils and students in discourse with their teachers. Teachers make skilled and frequent use of indirectness to sustain the involvement and self-esteem of their pupils.

6 The linguistic framework introduced in Chapter 2 has considerable potential for pragmatic interpretation of transcripts of mathematics talk. Conversation analysis offers somewhat different analytical perspectives on such data, the potential of which is touched on but not fully explored in this thesis.

What began as a study of the mathematical thinking of primary school children has become a way of looking at ways of learning and styles of teaching through the particular perspective of vague language. What stands out from this pragmatic analysis is that vagueness is not a deficiency, but an essential ingredient of communicative competence in mathematical interaction. In a conjecturing atmosphere, vague language is the means by which to say what you want to say, and as much as you wish to commit yourself to, in the context in which you speak. In that sense, everyone who teaches and learns mathematics in social situations could benefit from learning how to recognise such vagueness, and how to use it.

FINAL REMARKS

I conclude with some reflections on the process of writing which I have been engaged in, and my awareness of my own development as researcher and teacher.

ON WRITING

The process of writing has created the conditions under which I could think most lucidly and most freely. A conventional perception is that one 'has' the idea before committing it to 'paper'. This has often been the case; the ideas 'come' as I travel alone in the car, or when I lie awake at night. But the very act of writing seems to channel the energy, to focus the concentration. Thus, a great deal of synthetic sense-making occurs whilst one is actually sitting to write. The word-processor allows the notion that these ideas are provisional, that the commitment (to RAM and disc) is not irrevocable. Indeed, that the products of one's sense-making are modifiable conjectures.

Others have commented on their awareness of the interaction between thought and writing. Describing his earliest experiences as a researcher, Jean Piaget wrote:

> But for lack of a laboratory and guidance [...] the only thing I could do was to theorize and write. I wrote even if it was only for myself, for I could not think without writing - but it had to be in a systematic fashion as if it were to be an article for publication. (1952a, p. 241)

The "as if it were" is a luxury that academics can rarely afford in the 1990s. Nevertheless, I found the incentive to publish parts of my research in advance of the assembly of the thesis entirely beneficial in a formative sense. As Dave Hewitt so succinctly puts it,

> I write in order to learn. (1988, p. 61)

Reflecting in her doctoral thesis on 'the researcher as writer', Rita Nolder wrote:

> Writing went hand-in-hand with analysis - my own perception is that the physical act of writing [...] actually stimulates the process of analysis. (1992, p. 136)

It is clear that the act of 'writing up' a thesis, or 'writing down' ideas, is not purely a communicative act, for the benefit of some supposed audience. It may also be a creative act, for the enlightenment of the writer. It was also for me, a selective and cathartic one.

The bulk of my writing-up took place between April and August 1995. It was intensive.

What I discovered, or re-discovered, during this quite pressured period of writing is the function of writing as a filter, a process of selection. There is a strange psychic phenomenon which seems to be a commonly-experienced result of intensive writing. This is heightened sensitivity to the issue under consideration. At times, it seemed that whatever I read in books and newspapers, things that I heard on the radio, things that people said - almost everything seemed to confirm, to illuminate, to link in with the matter that I was working on. On the whole, this is a good thing, but not necessarily so. It can be overwhelming, and confuse the task of selecting and editing. It is, I suppose, a kind of 'high', with the dual possibilities of fresh insight and poor judgement. Sometimes I wrote all day, suspecting but not yet *knowing* that nothing that I was producing that day would survive the tests (or the maxims) of quality and relevance, to make it to the final document. But I could not make that decision until it had been written.

This was particularly and painfully true of Chapter 2, which has little in common with the first draft. I became knowledgeable about philosophical aspects of vagueness which had no place in the thesis. At one point, sensing my dismay at the prospect of 'wasted' effort, my external supervisor, Margaret Deuchar offered me one of the gems of advice I received about thesis-writing, when she reassured me:

> "It's good to know some things that *aren't* in your thesis".

Writing performed for me another, therapeutic function. This thesis has been a long time coming. I don't like missing deadlines, especially those I suggest to myself, and this work is about 24 years overdue. At one time, it might have been an undertaking in Mathematical Logic. In 1980, I decided that it would be in Mathematics Education. In 1990, I started. In the meantime, my head became filled to capacity with the debris of things that might have been part of a thesis - not this one, but a thesis from a different time. One of the functions of writing was to purge myself of the ideas left over from all the unwritten theses; to see those ideas written down, and then to let them go.

Early in 1995, just before the intensive period of writing, I solicited any relevant advice that my friends and colleagues might care to offer me. Not much of it was memorable or particularly useful. Some words of a colleague, Chrissie Poulson, are an exception.

> "Your thesis doesn't have to be your last word on the subject."

This was a constant reassurance towards the end of the main writing period. The process of selection which I have described is also one of omission. Some of what is omitted is simply not good or not relevant; a great deal more is either not yet known or not yet ready to be written.

MARKED PERSONAL

The cliché says that one of the main outcomes of research is change to the researcher. I recognise in myself some professional integration and personal rehabilitation achieved through the preparation of this thesis, and conclude with some thoughts about that.

In 'writing up' this thesis, I ended up making more reference to my own work than I had intended. To some extent this self-reference furnishes a documentary trace of changes in myself. Some of these changes occurred over a long time, others are quite recent changes in my own interpretation of matters pertaining directly to this thesis. Underpinning and perhaps surmounting all else that I have learned in recent years is enhanced understanding of the way I teach, insight into how I might develop that way for myself - for my own satisfaction and enjoyment, as well as for the benefit of my students.

I began, in the preface, by identifying a period of time when my pedagogic beliefs were in flux. The outcome (by no means achieved *in vacuo*) was personal commitment to a teaching style which I characterised in terms of problem posing and solving, interaction in student pairs or small groups, formation and public articulation of conjectures by students, and class discussion. This is fine for courses on problem solving or mathematical processes; but my major self-imposed challenge and effort as a lecturer for the last decade has been to deploy this style for the teaching of *mainstream* mathematics topics. I have not always found it easy to communicate or rationalise my conviction or my style to my students except by example; it is important that I should, since they, in turn, will become teachers. It has even been difficult, at times, to share my conviction with some of my colleagues; they are not a captive audience for my example, such as it is.

René Thom has asserted that:

> whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics. (1973, p. 204)

I have discovered a rationale for my pedagogic conviction in my study for this thesis; principally, but by no means exclusively, in the work of Polya and Lakatos. This assists me in naming and understanding what I try to do. This knowledge may assist in the articulation of the theoretical underpinning of the way I aspire to teach. It clarifies and gives analytical substance to the fallibilist approach to teaching which I had begun to develop empirically and intuitively. If research is to result in the gaining of wisdom as well as the acquisition of knowledge, then that knowledge must be embedded in a philosophy. It is from such a bedrock of personal meaning that knowledge, and wisdom, perhaps, can be communicated and shared - in faith and lovingness.

FOOTNOTES

CHAPTER 0

0.1 Amazingly, amongst some 10 million words of scientific, epistemological and psychological prose (and, for good measure, a "philosophical novel" written at the age of 20), Piaget wrote very little autobiography, of which only 20 pages (Piaget, 1952a) are available in English.

0.2 In fact, Ginsburg is comparing the dimensions of three related 'protocol' procedures: talking aloud (without interviewer intervention), verbal clinical interview and revised clinical interview. He argues that clinical methods are best suited to the requirements of cognitive research involving children, and my list focuses on the characteristics of verbal clinical methods.

0.3 It is supremely ironic that the APU - the very unit which had the capability to monitor national 'standards' of pupil performance under the National Curriculum - was wound up by the British government in 1988.

0.4 The term 'frame' is used differently in discourse analysis, to mean an assimilative cognitive structure, something like Piaget's 'schema'. See Brown and Yule (1983, pp. 238-41).

0.5 This stronger framing was also pragmatically expedient, in that these particular interviews were conducted, not by me, but by an assistant. Even allowing for some training and a dry-run pilot study for this stage, the assistant lacked substantial experience in the art and technique of clinical interviewing, and could not be *expected* to make sound, instantaneous, judgements of a contingent nature. The fact that he frequently did was a bonus.

0.6 I should point out that this is not a theological statement, quite the contrary. Whilst human beings are capable of 'insight', any ultimate meaning of things is bound to be a mystery to finite intelligences. In other words, I do subscribe to St. Paul's belief that "we see through a glass, darkly". (1 Corinthians 13:12)

0.7 I take it as read that detachment is necessary in the *interpretation* of phenomena in the interest of scientific integrity. My point (and Atkinson's) here is that *analytical* detachment (i.e. in isolating salient components of data) must be consciously exercised by the observer who is a 'member of the tribe' which is the object of study.

0.8 Saran's phrase "spiral-like research process" is a confirming echo of some methodological ideas that I presented three years ago (7th November 1992) at a Mathematics Education Research Day at the Open University. Having described what I saw as some significant features of my data, I wrote "If, then, there is to be validation of the conjectures so generated, it must be as a consequence of more detailed analysis of the same (generative) data. This begins to suggest a spiral programme of generative and validating activity - rather like the evaluation-action-evaluation spiral which characterises action research".

0.9 At first I dismissed this interest in my contribution to the transcribed conversations, believing that study of *children* is the key to the improvement of mathematics education. I haven't changed my mind about this.

CHAPTER 1

1.1 The first task is adapted from ATM (1967, p. 52), having also been adopted by the Assessment of Performance Unit (Foxman *et al.*, 1982, pp. 102-111). It is an incridibly rich 'starter' for investigation, and I keep returning to it for work with students. The second task is more mainstream; see Fletcher, 1969, pp. 275-81.

1.2 Mathematicians are programmed to associate 'induction' with Proof by Mathematical Induction, but I am speaking of induction here as a scientist would, in relation to discovery or invention. As I noted in Chapter 0, Peirce uses the word 'abduction' with similar meaning. Despite my appreciation of Peirce's maverick genius, I shall not adopt his term.

1.3 Whewell's personification of the characters Induction and Deduction (like the characters of Bunyan's Pilgrim's Progress) is delightful; it is as though they are two characters inhabiting the mind of the scientist. It is gratifying to note, moreover, that, Whewell, Master of Trinity College, does not conform to the stereotype and make Induction female (illogical, intuitive, uncertain, apt to lead, to seduce her companion, capable of error) and Deduction male (logical, secure, the steadying influence on his partner). In Whewell's time, it was taken for granted - certainly in Cambridge - that only men "did" science anyway, so it is all the more surprising that both characters are female ("She bounds to the top" ... "solidity of her companion's footing").

1.4 The formulation [Q] is a standard way of presenting inductive inference, and clearly well-suited to questions such as: is every daffodil yellow? In which case F is the set of daffodils and G the set of all yellow things. (A subset such as the set of yellow flowers also suffices). If I inspect, say, ten daffodils (or even, with Wordsworth, a host of them), and find that each is indeed yellow, then I am likely to reason inductively that every daffodil is yellow. Whilst [Q] seems to embrace a great many mathematical questions and propositions, it is not immediately clear that every mathematical question which might be answered by inductive methodology is necessarily of the form [Q], which is posed in terms of class inclusion. Nor is it clear that every conjecture which arises from inductive methodology is of the form [Q] (or, to be precise, a claim that every F is indeed a G). It seems particularly important to see whether the conjectural products of paradigm investigations such as Tasks 1 and 2 are accommodated by the Q-formula. A possible solution is as follows. In the case of Task (investigation) 1, the product is a mapping $g: N \rightarrow N$; in fact, $g(n)=2^{n-1}$, but that is a detail. What is essential is the claim (conjecture):

[C] for all $n \varepsilon N$, the number of partitions of n is in fact g(n).

Now, let F be a set of integers, each of which is the numbers of partitions of some positive integer n. Formally, F={r(n): n∈N} where r(n) is the number of partitions of n. Further, let G be the image of N under the mapping g i.e. G=g(N) or {g(n):n∈N}. The inductive conjecture C *includes* the claim that every F is a G, but C is much more specific; given any member x of F, C not only asserts that x is to be found amongst the elements of G, it actually specifies *which* element of G is to be identified with x. This, of course, is what mappings do, and it does appear to be a particularly demanding form of inductive reasoning, both in conception and in confirmation.

Task 2 (Reflections) can more easily (at one level) be cast in the mould of [Q], taking F to be the set of all composites of ordered pairs of plane reflections (in intersecting lines) and G to be the set of all rotations. Specifying precisely *which* element of G is to be identified with any given element of F requires the definition of a function $t: L \times L \to E$ where L is the set of lines in the plane and E is the Euclidean group of plane isometries.

1.5 The function f(x) is said to be a generating function of the sequence a(n) if, in the power series expansion of f(x), the coefficient of x^n is a(n). Euler 'knew' the generating function for E(n)-O(n) only as an inductive conjecture, and makes much of this status in the memoir. He had inferred it himself inductively; it fell to others, later, to give a deductive demonstration, and to name it Euler's Theorem. A rare case, in Number Theory, of correct attribution! See, for example, Burn (1982, pp. 142-143). The Lemma concerning E(n), O(n), combined with some formal manipulation of generating functions, leads on to Euler's Identity (Niven and Zuckerman, 1980, p. 274):

$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - p(n-22) - p(n-26) + p(n-35) + p(n-40) - \ldots$ taking, if necessary, p(0) to be 1.

which now entirely resembles his "law" for $\sigma(n)$

CHAPTER 2

2.1 The point is actually more subtle than Freudenthal suggests, in that the encyclopedia entry is not guilty of inappropriate precision, but of non-standard use of 'rounders' - see Chapter 5. Freudenthal's intended point was nicely made in a gardening feature some years ago in the *Cambridge Evening News*, presumably 'updated' from an older source. Describing how to make some structure or other it advised: "Use a piece of wood about 7.62 cm wide". More recently, a *Boots* home-brew kit has instructed me to "add 907 grams (2 lb) of white, granulated sugar".

2.2 This view derives, of course, from the philosophy of Plato. Use of the name 'platonism' to describe a philosophy of mathematics is due to Paul Bernays (1935).

2.3 Hedges which make things *less* fuzzy include, for example, 'true', as in 'You're a true friend'. Such hedges appear to make no contribution to the study of vagueness: except that there are circumstances where one suspects that their use is to counter uncertainty.

2.4 Taken out of the context from which it arose, Lakoff's choice of the word 'fuzzy' in his definition of 'hedge' seems somewhat self-consciously colloquial in a careful and rigorous academic paper. The choice is, without doubt, a tribute to Zadeh who chose (somewhat self-consciously?) to use 'fuzzy' in preference to the philosophically standard 'vague'. Along the way Zadeh considered and rejected 'cloudy' (Kosko, 1994, p. 145). Evidently, in the definition of 'hedge', 'fuzzy' may be taken to be synonymous with 'vague'.

2.5 An example which comes to mind, deeply embedded in the language of Number Theory, is '*perfect* square', meaning the square of an integer. Since in non-analytical Number Theory one is only dealing with integers anyway, the hedge 'perfect' is superfluous. It fact, its inclusion regularly confuses students, who ask me whether a perfect square must also be a perfect number (equal to the sum of its proper divisors); in which case there would be no 'perfect' squares ...

2.6 For let j denote Jack and let R, H be the sets of Rich and Handsome people respectively. Then, given that $m(R, j) = 0.7$ and $m(H, j) = 0.4$, it follows that $m(R', j) = 1 - m(R, j) = 0.3$. Hence $m(R' \cup H) = \max\{0.3, 0.4\} = 0.4$. Now $R \cap H' = (R' \cup H)'$, and so $m(R \cap H') = 1 - (R' \cup H) = 1 - 0.4 = 0.6$.

2.7 Briefly, the paradox asks the question: if single grains are removed one at a time from a heap of sand, at what point is it no longer a heap? An alternative formulation concerns a hairy man losing single hairs from his head; when is the man bald? The sorites paradox is important insofar as it appears to have identified, for western thought, a philosophical issue which potentially permeates all of life. A resurgence of interest in the sorites in modern times is evident (Russell, 1923; Goguen, 1969; Rolf, 1981; Sperber and Wilson, 1986b; Burns, 1991; Kosko, 1994; Williamson, 1994).

2.8 The expression derives from Bar Hillel's caveat (1971, p. 405) "Be careful with forcing bits and pieces you find in the pragmatic wastebasket into your favorite syntactico-semantic theory. It would perhaps be preferable to first bring some order into the contents of this wastebasket".

CHAPTER 3

3.1 The reference I:n.m indicates that, whilst the suggested utterance is intended to be plausible, it is invented (the mth such in Chapter n) for the purpose of exposition or clarification. The issue of the validity of such intuitive data is discussed later in this chapter.

3.2 Anaphora and deixis are not necessarily mutually exclusive: Levinson (1983, p. 67) gives the example

I was born in London and have lived there ever since.

in which 'there' is co-referential with 'London' (anaphora) but also locates (deixis) the utterance outside London. The deictic effect is achieved by the tacit choice of 'there' rather than 'here'.

3.3 I adopt a linguists' convention to use capital letters for Quality, etc. to mark the maxims. The esteem in which Grice's theory of implicature is held is indicated, in my view, by a number of attempts to revise an 'improve' it by proposing more economical alternatives. In particular, Horn (1984) has just two principles: 'Q' comprises Quality with the 'not too little' component of Quantity, and 'R' is 'not the too much' part of Quantity with Manner and Relevance. Sperber and Wilson (1986a) are more radical, proposing that a super-maxim of Relevance, along with a concept of 'mutual manifestness', subsumes them all. I judge that what Grice's original formulation lacks in economy is compensated for by its transparency, and prefer to adhere to it for my exposition here.

CHAPTER 4

4.1 The role of the pronoun 'it' was the sub-plot of a paper (Rowland, 1991) given at the first British Colloquium for Mathematics Education in Loughborough, and later the major theme of an article (Rowland, 1992a). This chapter is a revision and extension of that article.

4.2 This and all subsequent references to extracts from transcripts is by (a) transcript code, e.g. S2 for the second interview with Susie (see the list of transcripts in Appendix 1) and, where appropriate (b) speaker 'turn' (e.g. 32), or a range of turns (e.g. 23-42).

4.3 On the other hand, Wales (1980, p. 33) accounts for the prevalence of 'we' in scientific discourse in terms of the egocentric force of pronouns in English. Thus, the choice of 'we' in preference to 'I' is made in order to achieve rhetorical distancing of the speaker/writer from the content of what s/he says/writes, to achieve muted egocentricity.

4.4 The value of counting particular words in transcripts is limited, but electronic text storage now makes such counts both possible and relatively quick.

4.5 A more recent corpus, but not yet analysed in the way that I need, is that collected by Robin Fawcett and Michael Perkins in South Wales in the late 1970's, published in Fawcett and Perkins (1980). It consists of 65,000 words in 184 files, involving 92 children aged between 6 and 12, electronic versions of which have recently become available. Each child was recorded once in a play session and once in an interview. There could be considerable potential in a project which sifts the Fawcett and Perkins corpus for vocabulary and aspects of vague language, since it would provide a useful baseline for comparison.

4.6 The 'popularity index' used by Edwards and Gibbon is defined to be the product of the percentage of children in the sample using the word and the average use per child in the sample e.g. if precisely half of the children in the sample used a particular word, and each of them used it just once, the FI would be 25 (50 x 0.5). At age 7+ the highest popularity indices were found to be 3667 ('and'), 2968 ('the'), 2349 ('a'), 2330 ('I'), 1915 ('to'), 943 ('it'), 913 ('is'), 889 ('my'), 842 ('go').

For future reference, note that 'you' has index 102 at 7+ and is ranked 53rd. Not surprisingly there is a meteoric rise in the *use* of 'you' from 5 + (index 2.7) to 6+ (51) to 7+ (102); the ascent of 'you' in the *rankings* over the same period is less marked (122nd to 56th to 53rd). Whilst Howes does not give a ranking for 'you', a cursory inspection of his (alphabetical) word frequency list suggests that it is in the first ten for adult spoken language.

4.7 For comparison, I have consulted adult language data from the Lancaster-Oslo-Bergen (LOB) project (Hofland and Johansson, 1982) and Howes (1966). The LOB corpus is about 1 million words of written language from 15 categories of British sources (fiction, newspapers etc), all published in 1961. Howes' corpus consists of 250,000 words spoken by american adults. The relative frequency and ranking *in the LOB corpus* of the thirty relevant words in the LOB corpus is shown in Table 4.2. Howes does not list rankings, but relative frequencies are shown in the table together with rankings of the thirty words *among themselves.*

As I have remarked, the LOB and Howes data are from adult sources and must be used with caution in the present context. In fact (Table 4.2) they support the general conclusion that one would not expect 'it' to occur nearly as frequently as 'the', 'and' and 'to' (ranked 1, 2 and 4 in both sources). Howes also indicates that 'I' is generally much more prevalent than 'it' in (adult) spoken language, although not surprisingly the reverse is the case in adult writing (LOB).

4.8 Susie's use the deictic 'it' to refer to things she cannot name has a parallel in concern for proprriety in adult social practice, giving rise to deictic reference to semi-taboo topics, c.f. the car sticker "Windsurfers do it standing up", and numerous variants. The *double entendre* is achieved by the exploitation of vagueness.

4.9 Susie's use of 'real' to mean 'whole' is quite a discussion starter. It is certainly reminiscent of Kronecker's famous remark ("Gog made integers, all else is the work of man"), and calls to mind her comments about "the mathematics" being invented.

4.10 Quoted from the second iteration (DES, 1991) of the Mathematics National Curriculum for England and Wales. The third, post-Dearing version (DFE, 1995) has in its Level 5 description for Number and Algebra (p. 33): "They check their solutions by applying inverse operations or estimating ...".

4.11 Vestiges of the singular 'thou' survive in speech in northern England, and non-standard English includes plural forms of 'you' in various dialects e.g. 'y'all' (USA) and 'you'se' (Ireland).

CHAPTER 5

5.1 Pimm (1987, p. 52) observes that "transcripts of actual classes regularly indicate little verbal interaction between pupils themselves (particularly about mathematics)". I had in mind that this frequently *is* the case; this is not to accept that it *has to be* the case, but to recognise that it was not my aim (in this research) to attempt to change classroom culture.

5.2 I intend the word 'gambit', as I use it in this chapter, to mean little more than an opening move. Each of these interviews cannot, however, be viewed as a contest, in the way that chess is. On the other hand, I am deliberately setting out to manoeuvre the children into situations where they make predictions and generalisations. Insofar as they may be reluctant to do so, it could be seen as a sort of contest. There is imbalance of strength on both 'sides' - only I know the purpose of the game, but I am at their mercy in that they have the power to give or withhold the cognitive and linguistic behaviours I am setting out to provoke. Pimm (1987) associates the word 'gambit', in teacher questioning, with the possibility of *sacrifice*. In my case, this is appropriate to the extent that, in allowing the children some measure of control (over interpretation of the offered task), I may lose control of the direction of the interview in terms of engagement with certain mathematical processes.

5.3 Let n be a positive integer and f(n) be the number of pairs (a,b) such that a+b=n, where a, b belong to a set A of 'numbers'. If (b,a) is taken to be distinct from (a,b) (unless a=b) and A is the set N = {1,2,3, ...} of natural numbers, then f(n)=n-1; if A also includes zero then f(n)=n+1. If, however, (a,b) is always identified with (b,a), and A=N, then f(n)=½n when n is even, and ½(n-1) when n is odd. With zero included in A these become ½n+1 and ½(n+1) respectively. Of course, if A includes the set of integers, then f(n) is not finite.

5.4 The use of a linguistic formula such as 'like, if you do' to refer to a general relation or a general process - in this instance additive commutativity, or symbolic reversal - by means of an instance of that relation/process, is commonplace. It is an instance of the power of the generic example (Mason and Pimm, 1984; Balacheff, 1988; see Chapter 1) to evoke well-founded confidence in a related generality. See also the discussion of 'you' in Chapter 4.

5.5 These three points are the outcome of personal reflection, but the first two turn out to be remarkably similar to paraphrases (Wierzbicka, 1976, p. 330) of 'well' as a 'hesitation noise'. Such behaviour need not be associated with uncertainty, and it is perhaps tempting to dismiss some hedging behaviour as nothing more than prevarication. On the other hand such a judgement may be precipitate, given the extensive analysis by linguists of the pragmatic function of members of the class of semantically-vacuous 'discourse markers' such as 'well', 'oh' and 'you know' (Brockway, 1981; Schiffrin, 1987).

5.6 Harry had recently transferred to the state-maintained school from a famous independent preparatory school. Transfer documents from the prep. school gave little indication of Harry's actual attainment, but did observe that "Harry is the only boy in his form who has not obtained his own copy of the *Odyssey*". This remark could be interpreted as a revelation of Harry's independence of thought and action, and of his willingness to take risks. In any case, such comments assist those who inhabit different cultures (including the majority of English people) to understand what really matters in the great English 'public' (meaning 'private') schools. Harry's apparent negligence with regard to the classics, unlike other manifestations of his independent spirit (or indolence), proved not to disadvantage him at his new school.

5.7 But not invariably. Here (noted December 1994) Mark is asking his mother, Judy, about a Christmas present for his grandfather:

Mark: How much do you think a Ruth Rendell book will be?

Judy: About four ninety-nine.

5.8 Children seem to latch onto their preferred hedges. Whilst 'basically' is rare in my data, see, for example, Maher *et al.* (1994, pp. 213-4), where the use of 'basically' by the boy Alan is striking, and very much consistent with the analysis I give for Simon.

5.9 A strict truth-conditional interpretation of the sentence I:5.16 would be to say that it makes a statement which is true provided the number of beans in the jar is at least ten. A standard pragmatic view, however (Levinson, 1983, p. 106) is that a person uttering I:5.16 *implicates* 'ten and no more' because the hearer expects adherence to the maxim of Quantity. In everyday discourse, the pragmatic interpretation is assumed; indeed I suggest that the truth-conditional interpretation would be considered rude or ostentatiously 'clever'. There is an analogy with the truth-conditional interpretation of the question "Would you like tea or coffee?" which admits the answer "Yes".

5.10 My choice of the name 'zone of conjectural neutrality' for this space between articulation and belief was (somewhat tongue-in-cheek) inspired by Vygotsky's term 'zone of proximal development' for the gap between what a learner can do alone and what s/he can achieve with 'expert' assistance. The two zones are quite different, of course. I note with interest, however, that Hewitt (1994, p. 64) gives the name 'neutral zone' to a region which contains the collection of sensory stimuli ("offerings") from which a student selects a subset for her/his attention.

CHAPTER 6

6.1 van den Brink (1984) argues that the linguistic component, which he calls 'acoustic counting', develops and functions in young children quite independently of any reference to objects. He shows, moreover, that acoustic counting is free of some of the constraints of conventional 'quantity counting'.

6.2 I am grateful to Heather Cooke for articulating this suggestion, at a research seminar on vague quantitative language, given by Joanna Channell at the Faculty of Mathematics, Open University, Milton Keynes, on 14th June 1993.

6.3 The use of actual Smarties was vetoed by the interviewer, on the grounds that Smarties are a product of Rowntrees, which is a division of Nestlé, which in turn is the subject of a boycott intended to reverse the company's promotion of baby milk products in the Third World.

6.4 In the pilot study these and other forms of the questions were trialled. Notably the form "How many sweets do you think ...?" was trialled, despite unease that it might prompt a bias towards "I think ..." responses. In the event, there was no evidence from the trial to suggest that it did have that effect. Nevertheless we made the decision to reject "... do you think ..." formulations; not to have done so would have invited accusation of a biased design, whether or not this would be justified. We also considered simply asking "How many sweets are there ..?", but rejected it without trial on the grounds that it came over as too direct, somewhat aggressive and "testing". Only later did we rationalise the guidance of our intuition in preference of presenting the question as an 'indirect speech act' - see Chapter 7 (Politeness, and Case1: Hazel).

6.5 The social and economic structure of Britain has significantly changed since the publication of Berger and Luckmann's account. There are certainly home environments to be found in Britain in the 1990s which do not offer the same secure and cosy base for primary socialisation as the stereotype described by Berger and Luckmann. Furthermore, the processes of secondary socialisation may begin before formal schooling in other environments such as a child minder's home or a playgroup. I also observe here that the role-distance which distinguishes secondary from primary socialisation, may be applied by the adolescent individual back to the home situation, requiring a re-definition or re-negotiation of their place within the home.

6.6 Reference to teachers as 'functionaries' is quite shocking. In many countries they are literally that in the sense that they are civil servants. Perhaps the epithet is now largely apposite and deserved in England and Wales, since the State has assumed such tight regulation of the government of schools and even of the curriculum. Teachers (like milkmen) have been cast in the role of delivers of a product. The protests of the teaching profession have been (with some notable exceptions) barely audible; silent compliance is what one would expect of functionaries with no personal stake in the process which they are employed to assist.

6.7 Walkerdine (1988, pp. 89-92) points out that the form of the pedagogic discourse between mothers and their children, and that between teachers and their pupils, is remarkably similar, and that both frequently use pseudo-questions. She argues, however (and I am obliged to curtail some profound exposition) that teachers are principally observers, operating within a testing regime, whereas mothers are participants in the activities which are the discourse contexts.

6.8 Again, a study of the Fawcett corpus (Fawcett and Perkins, 1980) might produce a more definitive account of age-related differences in the use of the Markers under consideration.

CHAPTER 7

7.1 Advanced Supplementary (AS) courses have about half the content of an Advanced level course, with the same academic rigour. The majority of students following the course at

Rachel's college are very able, and studying three full A-level courses in addition to the AS, but a few may choose AS mathematics because they doubt their ability to do well in a full A-level mathematics course.

7.2 Within his novella *Poor Koko* (Jonathan Cape, 1974), John Fowles considers a burglar's frequent use of 'right' as a statement-tag (as in the exchange "I have very little with me" - "Then you won't miss it, right?"). Fowles comments:

"It ['right'] may grammatically be more often an ellipsis for 'Is that right?' than for 'Am I right?' - but I am convinced that the psychological significance is always of the latter kind. It means in effect, I am not at all sure that I am right ... the thing it cannot mean is self-certainty".

7.3 This is an example of stylistic transformation of the basic grammatical structures of sentences, in this case called 'fronting a subordinate clause object' (Leech, Deuchar and Hoogenrad, 1982, p. 128).

7.4 Jonathan's proof proceeds as follows. There are p^2 ordered pairs of elements mod p; the cases p=4k+1 and p=4k+3 are then dealt with separately. In the first case there are (by Jonathan's first theorem) [NT4:156] $(p-1)^2$ pairs with x^2+y^2 *not* equal to zero. Moreover, the number of solutions of $x^2+y^2=n$ is the same for each n [NT4:157, proved in Episode 4], and n takes p-1 distinct non-zero values. Hence the number of solutions for each non-zero n must be p-1.

The case when p=4k+3 similarly concludes with dividing p^2-1 by p-1 to show that there are p+1 solutions for each n.

CHAPTER 8

8.1 The Chomskyan notions of linguistic competence and performance - the first associated with knowledge of a system of rules, the second with the production of actual utterances - are loosely related to the Saussurian terms *langue* and *parole*. The distinction is complicated by the subsequent introduction of the term 'communicative competence' (Bernstein, 1971b, p. 146), which has a contextual dimension, and subsumes social aspects of language behaviour and awareness of communicative constraints. Further, a notion of 'pragmatic competence' has also been proposed. 'Communicative competence' seems to presuppose some awareness in the speaker, albeit tacit, of how to use language for a given purpose in a given context, and that is what I intend the term to mean when I use it.

8.2 Sfard's process-object formulation also nicely embraces the notion of 'it' as a conceptual deictic, when she writes:

Seeing a mathematical entity as an object means *being capable of referring to it* as if it was a real thing (Sfard, 1991, p. 4, emphasis added).

REFERENCES

Ackermann, W. (1956) 'Begründung einer strengen Implikation', *Journal of Symbolic Logic,* **21**, pp. 113-128.

Ainley, J. (1988) 'Perceptions of teachers' questioning styles', in Bourbàs, A. (ed.), *Proceedings of the Twelfth International Conference for the Psychology of Mathematics Education* Vol.1, Vezprém, Hungary, Ferenc Genzwein, OOK, pp. 92-99.

Ainley, J. (1991) 'Is there any mathematics in measurement?', in Pimm, D. and Love, E. (eds), *Teaching and Learning School Mathematics,* London, Hodder and Stoughton, pp. 69-76.

Atkinson, J. M. and Drew, P. (1979) *Order in Court,* London, Macmillan.

Atkinson, P. (1981) 'Inspecting classroom talk', in Adelman, C. (ed.), *Uttering, Muttering,* London, Grant McIntyre, pp. 98-114.

ATM (1967) *Notes on Mathematics in Primary Schools,* Cambridge, Cambridge University Press.

ATM (1980) *Language and Mathematics,* Nelson, Association of Teachers of Mathematics.

Austin, J. L. (1962a) *Sense and Sensibilia,* Warnock, G. J. (ed.), Oxford, Oxford University Press.

Austin, J. L. (1962b) *How to Do Things with Words,* Oxford, Oxford University Press.

Balacheff, N. (1988) 'Aspects of proof in pupils' practice of school mathematics', in Pimm, D. (ed.), *Mathematics, Teachers and Children,* London, Hodder and Stoughton, pp. 216-235.

Balacheff, N. (1990) 'Towards a *problématique* for research on mathematics teaching'. *Journal for Research in Mathematics Education,* **21**(4) pp. 258-272.

Bar-Hillel, Y. (1971) 'Out of the pragmatic wastebasket', *Linguistic Enquiry,* **2**, pp. 401-7.

Barham, J. C. (1988) *Teaching Mathematics to Deaf Children,* Unpublished Ph.D. thesis, University of Cambridge.

Bell, A.W., Costello, J. and Küchemann, D. (1983) *A Review of Research in Mathematical Education, Part A: research on learning and teaching,* Windsor, NFER-Nelson.

Berger, P. and Luckmann, T. (1967) *The Social Construction of Reality*, Harmondsworth, Penguin.

Bernays, P. (1935) 'Sur le Platonisme dans les Mathématiques', *L'Enseignement Mathématique 34*, reprinted in translation in Benacerraf, P. and Putnam, H. (eds), (1964) *Philosophy of Mathematics*, Oxford, Basil Blackwell, pp. 274-286.

Bernstein, B. (1971a) 'On the classification and framing of educational knowledge', in Young, M. F. D. (ed.), *Knowledge and Control: new directions for the sociology of education*, London, Collier-Macmillan, pp. 47-69.

Bernstein, B. (1971b) *Class, Codes and Control, Volume 1*, London, Routledge and Kegan Paul.

Bernstein, R. (1983) *Beyond Objectivism and Relativism*, Philadelphia, PA, University of Pennsylvania Press.

Bishop, A. (1985) 'The social construction of meaning - a significant development for mathematics education?', *For the Learning of Mathematics* 5(1) pp. 24-28.

Black, M. (1937) 'Vagueness: an exercise in logical analysis', *Philosophy of Science 4*, pp. 427-55.

Brenner, M. (1985) 'Intensive interviewing', in Brenner, M, Brown, J. and Canter, D. (eds), *The Research Interview: uses and approaches*, London, Academic Press, pp. 147-162

Bright, G. W. (1979) 'Estimating physical measurements', *School Science and Mathematics*, **79**(8), pp. 581-6.

Brockway, D. (1981) 'Semantic constraints on relevance', in Parrett, H., Sbisa, M. & Verschueren, J. (eds), *Possibilities and Limitations of Pragmatics: Proceedings of the Conference on Pragmatics at Urbino*, July 8-14, 1979, Amsterdam, Benjamins, pp. 57-78.

Brown, G. and Yule, G. (1983) *Discourse Analysis*, Cambridge, Cambridge University Press.

Brown, P, and Levinson, S. C. (1987) *Politeness: some universals in language usage*, Cambridge, Cambridge University Press.

Brown, R. W. and Gilman, A. (1970) 'Pronouns of power and solidarity', in Brown, R. (ed.), *Psycholinguistics*, New York, Free Press, pp. 302-336.

Brown, S. (1981) 'Sharon's 'Kye'', *Mathematics Teaching*, **94**, pp. 11-18.

Burgess, R. G. (1985) 'Introduction', in Burgess, R. G. (ed.), *Strategies of Educational Research: qualitative methods*, London, The Falmer Press, pp. 1-22.

Burn, R. P. (1982) *A Pathway into Number Theory*, Cambridge, Cambridge University Press.

Burns, L. C. (1991) *Vagueness: an investigation into natural languages and the sorites paradox*, Dordrecht, Kluwer Accademic Publishers.

Burroughs, G. E. R. (1957) *A Study of the Vocabulary of Young Children*, University of Birmingham, Institute of Education, Educational Monographs no. 1, Edinburgh, Oliver & Boyd.

Burton, L. (1984) *Thinking Things Through: problem solving in mathematics*, Oxford, Basil Blackwell.

Buxton, L. (1981) *Do You Panic about Mathematics?*, London, Heinemann.

Carlson, L. (1984) *'Well' in Dialogue Games: a discourse analysis of the interjection 'well' in idealised conversation*, Amsterdam, John Benjamins.

Chafe, W. (1972) 'Discourse structure and human knowledge' in Carroll, J. B. and Freedle, R. O. (eds), *Language Comprehension and the Acquisition of Knowledge*, Washington DC, Winston.

Champagne, A. B., Klopfer, L. E. and Anderson, J. H. (1980) 'Factors influencing the learning of classical mechanics', *American Journal of Physics*, **48**. pp. 1074-79.

Channell, J. M. (1980) 'More on approximations', *Journal of Pragmatics*, **4**, 461-476.

Channell, J. M. (1985) 'Vagueness as a conversational strategy', *Nottingham Linguistic Circular*, **14**, pp. 3-24.

Channell, J. M. (1990) 'Precise and vague quantities in writing in economics', in Nash, W. (ed.), *The Writing Scholar: studies in the language and conventions of academic discourse*, Beverley Hills, CA, Sage Publications, pp. 95-117.

Channell, J. M. (1994) *Vague Language*, Oxford, Oxford University Press.

Chevallard, Y. (1985) *La Transposition Didactique*, Grenoble, La Pensée Sauvage.

Christie, A. (1977) *Curtain: Poirot's Last Case*, London, Fontana Books.

Clayton, J. G. (1988) 'Estimation', *Mathematics Teaching*, 125 pp. 18-19.

Clayton, J. G. (1992) *Estimation in Schools*, Unpublished M.Phil. thesis, London, University of London,.

Coates, J. (1983) *The Semantics of the Modal Auxiliaries*, London, Croom Helm.

Cobb, P., Yackel, E. and Wood, T. (1992) 'Interaction and learning in mathematics classroom situations', *Educational Studies in Mathematics*, **23**, pp. 99-122.

Cornu, B. (1991) 'Limits', in Tall, D. (ed.), *Advanced Mathematical Thinking*, Dordrecht, Kluwer Academic Publishers.

Crystal, D. (1969) *Prosodic Systems and Intonation in English*, Cambridge, Cambridge University Press.

Crystal, D. (1991) *A Dictionary of Linguistics and Phonetics* (3rd Edition), Oxford, Basil Blackwell.

Cupitt, D. (1994) 'God and the regulative ideal', Letter to *The Independent*, 11th August, p. 20.

D.E.S. (1991) *Mathematics in the National Curriculum*, London, H.M.S.O.

D.F.E. (1995) *Mathematics in the National Curriculum*, London, H.M.S.O.

Dawson, S. (1991) 'Learning mathematics does not (necessarily) mean constructing the right knowledge', in Pimm, D. and Love, E. (eds), *Teaching and Learning School Mathematics*, London Hodder and Stoughton, pp. 195-204.

Delderfield, R. F. (1969) *Come Home Charlie, and Face Them*, London, Hodder and Stoughton.

Dewey, J. (1923) 'The pragmatism of Peirce', a supplementary essay in Peirce, C. S. (1923) pp. 301-308.

Donaldson, M. (1978) *Children's Minds*, London, Fontana.

Drew, P. (1981) 'Adults' corrections of children's mistakes', in French, P. and MacLure, M. (eds), *Adult-Child Conversation*, London, Croom Helm, pp. 244-268.

Dubois, B. L. (1987) '"Something on the order of around forty to forty-four": imprecise numerical expressions in biomedical slide talks', *Language in Society*, **16**, pp. 527-41.

Dubois, J. (1969) *Grammaire structurale du français: La phrase et les transformations*, Paris, Larousse.

Durkin, K. and Shire, B. (1991) 'Lexical ambiguity in mathematics contexts', in Durkin, K. and Shire, B. (eds), *Language in Mathematical Education*, Milton Keynes, Open University Press, pp. 71-85.

Edwards, R. P. A. and Gibbon, V. (1973) *Words Your Children Use,* London & Toronto, Burke Books.

Eisenhart, M. A. (1988) 'The Ethnographic Research Tradition and Mathematics Education Research', *Journal for Research in Mathematiics Education,* **19**(2), pp. 99-114.

Drew, P. and Heritage, J. (1992) 'Analyzing talk at work: an introduction', in Drew, P. and Heritage, J. (eds) *Talk at work: interaction in institutional settings,* Cambridge, Cambridge University Press, pp. 3-63.

Ellis, B. (1968) *Basic Concepts of Measurement,* Cambridge, Cambridge University Press.

Ernest, P. (1983) 'Thinking of a funny line', *Mathematical Education for Teaching,* **4**(2), pp. 30-40

Ernest, P. (1991) *The Philosophy of Mathematics Education,* London, The Falmer Press.

Farnham, D. (1975) 'Language and mathematical understanding', *Recognitions,* **3**, Derby, Association of Teachers of Mathematics.

Fauvel, J. (1987) *The Greek Concept of Proof Block 1, Unit 3, OU course MA290,* Milton Keynes, Open University Press.

Fawcett R. P. and Perkins M. R. (1980) *Child Language Transcripts 6 - 12, with a preface,* Pontypridd, Polytechnic of Wales, Dept of Behavioural and Communication Studies.

Fielker, D. (1988) 'Metaphors and models', *Mathematics Teaching,* **124**, pp. 4-6.

Fielker, D. (1993) 'Big numbers for small minds', *Times Educational Supplement,* 8th October 1993, p. XXVIII.

Fisch, M. H. and Turquette, A. R. (1966) 'Peirce's triadic logic', *Transactions of The Charles S. Peirce Society,* ***2***, pp. 71-85.

Fischbein, E. (1987) *Intuition in Science and Mathematics: an educational approach,* Dordrecht, D. Reidel.

Fischbein, E. (1990) 'Introduction', in Nesher, P. and Kilpatrick, J. (eds), *Mathematics and Cognition: a research synthesis by the International Group for the Psychology of Mathematics Education,* Cambridge, Cambridge University Press, pp. 1-13.

Fletcher, T.J. (ed.), (1969) *Some Lessons in Mathematics,* Cambridge, Cambridge University Press.

Foxman, D. D., Cresswell, M. J. and Badger, M. E. (1981) *Mathematical Development: primary survey report no. 2,* London, H.M.S.O.

Foxman, D. D., Cresswell, M. J., Ward, M., Badger, M. E., Tuson, J. A. and Bloomfield, B. A. (1980) *Mathematical Development: primary survey report no. 1,* London, H.M.S.O.

Foxman, D. D., Ruddock, G J., Badger, M. E. and Martini, R.M. (1982) *Mathematical Development: primary survey report no. 3,* London, Her Majesty's Stationary Office.

Freudenthal, H. (1978) *Weeding and Sowing: preface to a science of mathematical education,* Dordrecht, D. Reidel.

Freudenthal, H. (1991) *Revisiting Mathematics Education: the China lectures,* Dordrecht, Kluwer Academic Publishers.

Fuson K. C. (1988) *Chidren's Counting and Concepts of Number,* New York, Springer Verlag.

Fuson, K. C. (1991) 'Children's early counting: saying the number-word sequence, counting objects, and understanding cardinality', in Durkin, K. and Shire, B. (eds), *Language in Mathematical Education,* Milton Keynes, Open University Press, pp. 27-39.

Garfinkel, H. and Sacks, H. (1970) 'On formal structures of practical actions', in McKinney, C. and Tiriakian, E. (eds), *Theoretical Sociology,* New York, Appleton Century Crofts. pp. 337-366.

Garrod, S, C. and Sanford, A. J. (1982) 'The mental representation of discourse in a focused memory system', *Journal of Semantics,* **1**, pp. 21-41.

Gattegno, G. (1981) 'Children and Mathematics: a new appraisal', *Mathematics Teaching,* **94** pp. 5-7.

Gazdar, G. (1979) *Pragmatics: implicature, presupposition and logical form,* New York, Academic Press.

Gelman, R. (1977) 'How young children reason about small numbers', in Castellan, N.J., Pisoni, D.B. and Potts, G.R. (eds), *Cognitive Theory Vol. 2,* Hillsdale, New Jersey, Lawrence Erlbaum Associates, pp. 219-238.

Gelman, R., and Gallistel, C. R. (1978) *The Child's Understanding of Number,* Cambridge, Mass., Harvard University Press.

Giarelli, J. B. (1988) 'Qualitative inquiry in philosophy and education: notes on a pragmatic tradition', in Sherman, R. R. and Webb, R. B. (eds), *Qualitative Research in Education: focus and methods,* London, The Falmer Press, pp. 22-27.

Ginsburg, H. (1981) 'The clinical interview in psychological research on mathematical thinking: aims, rationales, techniques', *For the Learning of Mathematics*, **1**(3) pp. 4-11.

Frankel, R. (1990) 'Talking in interviews: a dispreference for patient-initiated questions in physician-patient interviews', in Psathas, G. (ed.) *Studies in Ethnomethodology and Conversation Analysis*, Lanham Md, University Press of America, pp. 231-262.

Ginsburg, H. and Russell, R. L. (1981) 'Social class and racial influences on early mathematical thinking', *Monographs of the Society for Research in Child Development*, **46**(6), pp. 32-51.

Ginsburg, H. P. (1977) *Children's Arithmetic: the learning process*, New York, D. Van Nostrand Co.

Ginsburg, H. P., Kossan, N. E., Schwartz, R. and Swanson, D. (1983) 'Protocol methods in research on mathematical thinking', in Ginsburg, H. B. (ed.), *The Development of Mathematical Thinking*, New York Academic Press, pp. 7-47.

Glaser, B. G. and Strauss, A. L. (1967) *The Discovery of Grounded Theory: strategies for qualitative research*, Chicago, Aldine Publishing Co.

Goffman, E. (1967) *Interaction Ritual: essays on face to face behavior*, New York, Doubleday Anchor.

Goguen, J. A. (1969) 'The logic of inexact concepts', *Synthese*, **19**. pp. 325-373.

Goody, E. (ed.), (1978) *Questions and Politeness: strategies in social interaction*, Cambridge, Cambridge University Press.

Gordon, D. and Lakoff, G. (1971) 'Conventional postulates', *Papers from the Seventh Regional Meeting of the Chicago Linguistics Society*, Chicago, Chicago Linguistic Society, pp. 63-84.

Graham, D. (1993) *A Lesson for us all: the making of the National Curriculum*, London, Routledge.

Granville, G. and Katz, I. (1993) 'The number's up for maths' greatest riddle', *The Guardian*, June 24th, Section 2, pp. 2-3.

Grice, H. P. (1975) 'Logic and conversation', in Cole, P. and Morgan, J. L. (eds), *Syntax and Semantics: Vol. 3 Speech Acts*, New York, Academic Press, pp. 41-58.

Grice, H. P. (1989) *Studies in the Way of Words*, Cambridge, Mass., Harvard University Press.

Halliday, M. A., McIntosh, A. and Stevens, P. (1964) *The Linguistic Sciences and Language Teaching*, London, Longmans.

Hardcastle, L. and Orton, T. (1993) 'Do they know what we are talking about?', *Mathematics in School*, **22**(3), pp. 12-14.

Hardy, G. H. (1940) *A Mathematician's Apology*, Cambridge, Cambridge University Press. (Reprinted 1967 with a foreward by C. P. Snow.)

Harel, G. and Tall, D. (1991) *The general, the abstract, and the generic in advanced mathematics. For the Learning of Mathematics*, **11**(1), pp. 38-42.

Hart, K. M. (1979) 'Ratio and proportion', in Hart, K. M. (ed.), *Children's Understanding of Mathematics 11-16*, London, John Murray, pp. 88-102.

Hempel, C. G. (1965) *Aspects of Scientific Explanation*. New York, The Free Press.

Hersh, R. (1979) 'Some proposals for reviving the philosophy of mathematics', *Advances in Mathematics*, **31**, pp. 31-50.

Hersh, R. (1993) 'Proving is convincing and explaining', *Educational Studies in Mathematics*, **24**, pp. 389-399.

Hewitt, D. (1988) 'Postscript', *Mathematics Teaching*, **123**, p. 61.

Hewitt, D. (1992) 'Train spotters' paradise', *Mathematics Teaching*, **140**, pp. 6-8.

Hewitt, D. (1994) *The Principle of Economy in the Learning and Teaching of Mathematics*, Unpublished Ph.D. thesis. Milton Keynes, The Open University.

Heyting, A. (1964) 'The intuitionistic foundations of mathematics', in Benacerraf, P. and Putnam, H. (eds), (1964) *Philosophy of Mathematics*, Oxford, Basil Blackwell, pp. 42-49.

H.M.S.O. (1982) *Mathematics Counts*, London, H.M.S.O.

Hofland, K. and Johansson. S. (1982) *Word Frequencies in British and American English*, Oslo, The Norwegian Computing Centre for the Humanities.

Holland, J. H., Holyoak, K. J, Nisbett, R. E, and Thagard, P. R. (1986) *Induction: processes of inference, learning and discovery*, Cambridge, MA, MIT Press.

Holt, J. (1969) *How Children Fail*, Harmondsworth, Penguin Books.

Horn, L. (1984) 'Toward a new taxonomy for pragmatic inference', in Schiffrin, D. (ed.), *Georgetown Round Table on Languages and Linguistics* 1984, New York, Georgetown University Press, pp. 11-42.

Howes, D. (1966) 'A word count of spoken English', *Journal of Verbal Learning and Verbal Behaviour*, **5**, pp. 572-604.

Hymes, D. (1972) 'Introduction', in Cazden, C. B., John, V. P. and Hymes, D. (eds) *Functions of Language in the Classroom*, New York, Teachers College Press, pp. xi-lvii.

Heritage, J. (1984) *Garfinkel and Ethnomethodology*, Cambridge, Polity Press.

Hopper, R. (1989) 'Conversation analysis and social psychology as descriptions of interpersonal communication', in Roger, D. and Bull, P. (eds) *Conversation: an interdisciplinary perspective*, Clevedon, Multilingual Matters, pp. 48-65.

Inhelder, B. and Piaget, J. (1958) *The Growth of Logical Thinking from Childhood to Adolescence*, New York, Basic Books.

Jensen, E. M., Reese, E. P. and Reese, T. W. (1950) 'The subitising and counting of visually presented fields of dots', *The Journal of Psychology*, **30**, pp. 363-392.

Joffe, L. (undated, *circa* 1985) *Practical Testing in Mathematics at Age 11*, London, Assessment of Performance Unit.

Johnson, P. (1970) 'A diiferent value system and its effect on teaching methods', *Mathematics Teaching*, **53**, pp. 50-52.

Kane, R. B., Byrne, M. A., Hater, M. A. (1974) *Helping Children Read Mathematics*, New York, American Book Company.

Keats, J. (1820) 'Ode to a Grecian Urn', in *Lamia, Isabella, the Eve of St Agnes and other poems*, London, Taylor and Hessey.

Kilpatrick, J. (1988) 'Editorial', *Journal for Research in Mathematics Education* **19**(2), p. 98.

Kosko, B. (1994) *Fuzzy Thinking: the new science of fuzzy logic*, London, HarperCollins.

Laberge, S. and Sankoff, G. (1980) 'Anything you can do', in Sankoff, G. (ed.), *The Social Life of Language*, Philadelphia, University of Pennsylvania Press, pp. 271-294.

Laborde, C. (1979) 'Audacity and reason: French research in mathematics education', *For the Learning of Mathematics*, **9**(3), pp. 31-36.

Labov, W. (1970) 'The study of language in its social context', *Studium Generale*, **23**, pp. 30-87.

Labov. W. and Fanshel, D. (1977) *Therapeutic Discourse: psychotherapy as conversation*, New York, Academic Press.

Lakatos, I. (1976) *Proofs and Refutations: the logic of mathematical discovery*, Cambridge, Cambridge University Press.

Lakoff, G. (1972) 'Hedges: a study in meaning criteria and the logic of fuzzy concepts', *Papers from the Eighth Regional Meeting of the Chicago Linguistics Society*, Chicago, Chicago Linguistic Society, pp. 183-228.

Lakoff, G. (1973) 'Hedges: A study in meaning criteria and the logic of fuzzy concepts', *Journal of Philosophical Logic* , **2**, pp. 458-508.

Lakoff, R. (1973) 'Questionable answers and answerable questions', in Kachru, B. B. *et al.* (eds), *Issues in Linguistics: Papers in Honor of Henry & Renée Kahane*, Urbana, University of Illinois Press, pp. 453-467.

Lakoff, R. (1975) *Language and Woman's Place*, New York, Harper.

Leech, G. (1983) *Principles of Pragmatics*, London, Longman.

Leech, G., Deuchar, M., Hoogenraad, R. (1982) *English Grammar for Today: a new introduction*, London, Macmillan.

Lerman, S. (1989) 'Constructivism, mathematics and mathematics education', *Educational Studies in Mathematics*, **20**, pp. 211-223.

Levinson, S. C. (1983) *Pragmatics*, Cambridge, Cambridge University Press.

Love, E. (1988) 'Evaluating mathematical activity', in Pimm, D. (ed.), *Mathematics, Teachers and Children*, London, Hodder and Stoughton, pp. 249-263.

Lukasiewicz, J. (1920) 'O logice trójwartosciowej', [On three-valued logic] *Ruch Filozoficzny* **5**, pp. 169-171.

Lunzer, E. (1978) 'Formal reasoning: a re-appraisal', in Presseisen,B. Goldstein, D. and Appel, M. H. (eds), *Topics in Cognitive Development, Vol. 2 of Language and Operational Thought*, New York, Plenum, pp. 47-77.

Lyons, J. (1968) *Introduction to Theoretical Linguistics*, Cambridge, Cambridge University Press.

Lyons, J. (1977) *Semantics*, Cambridge, Cambridge University Press.

Mackie, A. (1923) 'Psycho-analysis and education', *Australasian Journal of Psychology and Philosophy*, **1**, pp. 105-110.

MacLure, M. and French, P. (1980) 'Routes to right answers: on pupils', strategies for answering teachers', questions', in Woods, P. (ed.), *Pupil Strategies: explorations in the sociology of the school*, London, Croom Helm, pp. 74-93.

MacWhinney, B. (1991) *The CHILDES Project: Tools for Analyzing Talk*, Hillsdale NJ, Lawrence Erlbaum Associates.

Maher, C. A., Martino, M. M. and Davis, R. B. (1994) 'Children's different ways of thinking about fractions', in da Ponte, J. and Matos, J. F. (eds.) *Proceedings of PME-18, Volume 3*, Lisbon, University of Lisbon, pp. 208-15.

Mason, J. (1988) *Learning and Doing Mathematics*, London, Macmillan.

Mason, J. and Pimm, D. (1984) 'Generic examples: seeing the general in the particular', *Educational Studies in Mathematics*, **15**, pp. 277-289.

Mey, J. (1993) *Pragmatics*, Oxford, Basil Blackwell.

Mill, J. S. (1843) *A System of Logic Ratiocinative and Inductive: being a connected view of the principles of evidence and the methods of scientific investigation, 4th edition*, London, John W. Parker.

Morris, C. W. (1938) 'Foundations of the theory of signs', in Neurath, O., Carnap, R. and Morris, C. (eds), *International Encyclopaedia of Unified Science*, Chicago, University of Chicago Press.

Moxey, L. M. and Sanford, A. J. (1993) *Communicating Quantities: a psychological approach*, Hillsdale NJ, Laurence Erlbaum Associates.

Mühlhäusler, P. and Harré, R. (1990) *Pronouns and People: the linguistic construction of social and personal identity*, Oxford, Basil Blackwell.

Niven, I. and Zuckerman, H. S. (1980) *An Introduction to the Theory of Numbers* (4th edition), New York, John Wiley & Sons.

Nolder, R. (1992) *Bringing Teachers to the Centre of the Stage: A Study of Secondary Teachers' Responses to Curriculum Change in Mathematics*, Unpublished Ph.D. thesis, London, University of London.

Orton, A. (1985) *Studies in Mechanics Learning*, Leeds, University of Leeds Centre for Studies in Science and Mathematics Education.

Otterburn, M. K. and Nicholson, A. R. (1976) 'The language of CSE mathematics', *Mathematics in School*, **5**(5), pp. 18-20.

Paley, V. G. (1981) *Wally's Stories*, Cambridge, Mass. Harvard University Press.

Palmer, F. R. (1979) *Modality and the English Modals*, London, Longman.

Peirce, C. S. (1902) 'Vague', in Baldwin, J. M. (ed.), *Dictionary of Philosophy and Psychology*, London, Macmillan.

Peirce, C. S. (1923) *Chance, Love and Logic*, edited by Cohen, M. R., London, Kegan Paul, Trench, Trubner & Co, Ltd.

Peirce, C. S. (1932) *Collected Papers, Volume II: Elements of Logic*, edited by Hartshorne, C., Weiss, P. and Burks, A., Cambridge, Mass., Harvard University Press.

Peirce, C. S. (1934) *Collected Papers, Volume V: Pragmatism and Pragmaticism*, edited by Hartshorne, C., Weiss, P. and Burks, A., Cambridge, Mass., Harvard University Press.

Perera, K. (1990 'Grammatical differentiation between speech and writing in children aged 8 to 12', Carter, R. *Knowledge about Language*, London, Hodder and Stoughton, pp. 216-234.

Piaget, J. (1929) *The Child's Conception of the World*, New York, Harcourt, Brace and Company.

Piaget, J. (1952a) 'Jean Piaget', in Boring, E. G et al. (eds), *History of Psychology in Autobiography Vol. IV*, Worcester, Mass. Clark University Press, pp. 237-256.

Piaget, J. and Szeminska, A. (1952b) *The Child's Conception of Number*, London, Routledge and Kegan Paul.

Piérrault-Le Bonniec, G. (1980) *The Development of Modal Reasoning: genesis of necessity and possibility notions*, London, Academic Press.

Pimm, D. (1981) 'Metaphor and Analogy in Mathematics', *For the Learning of Mathematics*, 1(3) pp. 47-50.

Pimm, D. (1986) 'Beyond Reference', *Mathematics Teaching*, **116**, pp. 48-51.

Pimm, D. (1987) *Speaking Mathematically: communication in mathematics classrooms*, London, Routledge & Kegan Paul.

Pimm, D. (1992) 'Classroom language and the teaching of mathematics', in Nickson, M. and Lerman, S. (eds), *The Social Context of Mathematics Education: theory and practice*, London, South Bank University, pp. 67-81.

Pimm, D. (1994) 'Mathematics classroom language: form, function and force', in Biehler, R., Scholz, R., Straßer, R. and Winkelmann, B. (eds), *Didactics of Mathematics as a Scientific Discipline*, Dordrecht, Kluwer Academic Publishers, pp. 159-169.

Pirie, S. E. B., Martin, L. and Kieran, T. (1994) 'Mathematical images for fractions: help or hindrance?', in da Ponte, J. and Matos, J. F. (eds), *Proceedings of PME-18, Volume 3*, Lisbon, University of Lisbon, pp. 247-54.

Plunkett, S. (1979) 'Icons', *Mathematics Teaching*, **86**, pp. 6-7.

Polya, G. (1945) *How to Solve it: a new aspect of mathematical method*, Princeton, NJ., Princeton University Press.

Polya, G. (1954) *Mathematics and Plausible Reasoning, Volume 1: Induction and Analogy in Mathematics*, Princeton, NJ, Princeton University Press.

Polya, G. (1954) *Mathematics and Plausible Reasoning, Volume 2: Patterns of Plausible Inference*, Princeton, NJ, Princeton University Press.

Polya, G. (1962) *Mathematical Discovery: On understanding, learning and teaching problem solving. Volume I*, New York. John Wiley and Sons.

Polya, G. (1965) *Mathematical Discovery: On understanding, learning and teaching problem solving. Volume II*, New York, John Wiley and Sons.

Polya, G. (1992) Quoted in *For the Learning of Mathematics*, **12**(2), p. 11.

Post. E. (1941) 'Absolutely unsolvable problems and relatively undecidable propositions - account of an anticipation' in Davis, M. (1994) *Solvability, Provability, Decidability: the collected works of Emil L. Post*, Boston, MA, Birkhäuser, pp. 375-441

Presmeg, N. C. (1986) 'Visualisation in high school mathematics', *For the Learning of Mathematics*, **6**(3), pp. 42-46.

Prince, E. F., Frader, J. and Bosk, C. (1982) 'On hedging in physician-physician discourse', in di Pietro, R. J. (ed.), *Linguistics and the Professions*, Norwood, NJ, Ablex, pp. 83-96.

Rees, A. (1983) 'Pronouns of Person and Power', Unpublished M.A. Dissertation, University of Sheffield.

Rescher, N. (1979 *Cognitive Systematization: a systems-theoretic approach to a coherentist theory of knowledge*, Oxford, Basil Blackwell.

Rescher, N. (1980) *Induction: an essay on the justification of inductive reasoning*, Oxford, Basil Blackwell.

Rinsland, H.D. (1945) *A Basic Vocabulary of Elementary School Children*, New York, Macmillan.

Rolf, B. (1981) *Topics on Vagueness*, Lund, Sweden, Lunds Universitet.

Rorty, R. (1980) *Philosophy and the Mirror of Nature*, Oxford, Basil Blackwell.

Rosch, E. (1975) 'Cognitive reference points', *Cognitive Psychology*, **104**, pp. 192-233.

Rowland, T. (1974) 'Real functions which generate the dihedral groups', *Mathematics Teaching*, **69**, pp. 40-47.

Rowland, T. (1982) 'Teaching directed numbers: an experiment', *Mathematics in School*, **11**(1), pp. 24-27.

Rowland, T. (1990) 'Apparatus for number work', *CAN Newsletter 2*, Cambridge, CAN Continuation Project, Homerton College, pp. 1-2,

Rowland, T. (1991) 'Mathematical conversations with an articulate nine-year-old', Unpublished paper given at the First British Congress in Mathematics Education, Loughborough, July 1991.

Rowland, T. (1992a) 'Pointing with pronouns', *For the Learning of Mathematics* ***12***(2), pp. 44-48.

Rowland, T. (1992b) 'Pop maths postscript', *Mathematical Gazette* **76**(476), pp. 254-6.

Rowland, T. (1993) '"Hyperpainting" - mind games for night riders', *Mathematics Teaching*, **143**, pp. 28-30.

Rowland, T. (1994a) 'Conveying uncertainty in mathematics: towards a developmental account', Unpublished paper given at a conference of the British Educational Research Association, Oxford, September 1994.

Rowland, T. (1994b, second edition) *CAN in Suffolk: the first six months of a calculator-aware number curriculum*, Homerton Research Reports Series, Cambridge, Publication Unit, Homerton College Cambridge.

Rowland, T. (1995a) 'Between the lines: the languages of mathematics', in Anghileri, J. (ed.), *Children's Mathematical Thinking in the Primary Years*, London, Cassell, pp. 54-73.

Rowland, T. (1995b) 'Hedges in mathematics talk: linguistic pointers to uncertainty', *Educational Studies in Mathematics*, **29**(4), pp. 327-353.

Rowland, T. (1996) 'Researching mathematical thinking with talk, tape and transcripts', *Proceedings of The Third British Congress in Mathematics Education*, Manchester. Manchester Metropolitan University, pp. 279-286.

Russell, B. A. W. (1923) 'Vagueness', *Australasian Journal of Psychology and Philosophy*, **1**, pp. 84-92.

Sadock, J. M. (1977) 'Truth and approximations', *Berkeley Linguistic Society Papers*, **3**, pp. 430-439.

Saran, R. (1985) 'The use of archives and interviews in research on educational policy', in Burgess, R. G. (ed.), *Strategies of Educational Research: qualitative methods*, London, The Falmer Press, pp. 207-241.

Saxe, G. B. and Kaplan, R. (1981) 'Gesture in early counting: a developmental analysis', *Perceptual and motor skills*, **53**, pp. 851-854.

Schiffrin, D. (1987) *Discourse Markers*, Cambridge, Cambridge University Press.

Schiffrin, D. (1994) *Approaches to Discourse*, Oxford, Basil Blackwell.

Schumann, H. and Green, D. (1994) *Discoveing Geometry with a Computer - using Cabri Géomètre*, Bromley, Chartwell-Bratt.

Searle, J. R. (1969) *Speech Acts*, Cambridge, Cambridge University Press.

Searle, J. R. (1975) 'Indirect Speech Acts', in Cole, P. and Morgan, J. L. (eds), *Syntax and Semantics: Vol. 3 Speech Acts*, New York, Academic Press, pp. 59-82.

Semadeni, Z. (1984) 'Action proofs in primary mathematics teaching and in teacher training', *For the Learning of Mathematics*, *4*(1), pp. 32-34.

Sfard, A. (1991) 'On the dual nature of mathematical conceptions', *Educational Studies in Mathematics*, **22**, pp. 191-228.

Shuard, H. and Rothery, A. (1984) *Children Reading Mathematics*, London, John Murray.

Shuard, H. (1986) *Primary Mathematics Today and Tomorrow*, York, Longman/SCDC.

Siegel, W. S., Goldsmith, L. T. and Madson C. R. (1982) 'Skill in estimation problems of extent and numerosity', *Journal for Research in Mathematics Education*, ***13***(3), pp. 211-232.

Simons, H. (1981) 'Conversation Piece: the practice of interviewing in case study research', in Adelman, C. *Uttering, Muttering*, London, Grant McIntyre, pp. 27-51.

Sinclair, J. McH. and Coulthard, R. M. (1975) *Towards an Analysis of Discourse: the English used by teachers and pupils*, Oxford, Oxford University Press.

Skemp, R. R. (1979) *Intelligence, Learning and Action*, Chichester, John Wiley and Sons.

Sowder, J. (1992) 'Estimation and Number Sense', in Grouws, D. A. (ed.), *Handbook of Research on Mathematics Teaching and Learning*, New York, Macmillan Publishing Company, pp. 371-89.

Sperber, D. and Wilson, D. (1986a) *Relevance: communication and cognition*, Oxford, Basil Blackwell.

Sperber, D. and Wilson, D. (1986b) 'Loose talk', *Proceedings of the Aristotelian Society*, 86, Aristotelian Society, pp. 153-171.

Stamp, M. (undated, *circa* 1980) 'Perimeter equals area', in *Topics in Mathematics*, p. 3, Derby. Association of Teachers of Mathematics.

Steffe L. P. and Cobb P. (1988) *Construction of Arithmetical Meanings and Strategies*, New York, Springer Verlag.

Steffe L. P., von Glasersfeld, E., Richards, J. and Cobb P. (1983) *Children's Counting Types: Philosophy, Theory and Application*, New York, Praeger.

Steffe, L. P. (1991) 'The constructivist teaching experiment: illustrations and implications', von Glasersfeld, E. (ed.), *Radical Constructivism in Mathematics Education*, Dordrecht, Kluwer Academic Publishers, pp. 177-194,

Stephany, U. (1986) 'Modality', in Fletcher, P. and Garman, M. (eds), *Language Acquisition*, (second edition) Cambridge, Cambridge University Press.

Stern, G. (1964) *Meaning and Change of Meaning*, Bloomington, IN., Indiana University Press.

Stubbs, M. (1986) "A matter of prolonged fieldwork': notes towards a modal grammar of English', *Applied Linguistics*, 7(1), pp. 1-25.

Stubbs, M. (1996) Text and Corpus Linguistics, Oxford, Basil Blackwell.

Thom, R. (1973) 'Modern mathematics: does it exist?', in Howson, A., *Developments in Mathematics Education*, Cambridge, Cambridge University Press.

Tiegen, K. M. (1990) 'To be convincing or to be right: a question of preciseness', in Gilhooly, K. J., Keane, M. T. G., Logie, R. H. and Erdos, G. *Lines of Thinking: reflections on the psychology of thought*, Chichester, Wiley, pp. 299-314.

Tirosh, D. (1991) 'The role of students', intuitions of infinity in teaching the Cantorial theory', in Tall, D. (ed.), *Advanced Mathematical Thinking*, Dordrecht, Kluwer Academic Publishers, pp. 199-214,

Ullmann, S. (1962) *Semantics: an introduction to the science of meaning*, Oxford, Basil Blackwell.

van den Brink, J. (1984) 'Acoustic counting and quantity counting', *For the Learning of Mathematics*, 4(2), pp. 2-13.

von Glasersfeld, E. (1989) 'Constructivism in Education', in Husen, T. and Postlethwaite, T. N. (eds), *The International Encyclopedia of Education, Supplementary Volume*, Oxford, Pergamon Press.

Wales, K. M. (1980) 'Exophora re-examined: the uses of the personal pronoun WE in present-day English', *UEA Papers in Linguistics,* **12**, pp. 21-44.

Wales, R. (1986) 'Deixis', in Fletcher, P. and Garman, M. (eds), *Language Acquisition,* (second edition), Cambridge, Cambridge University Press, pp. 401-428.

Walkerdine, V. (1988) *The Mastery of Reason: cognitive development and the production of rationality*, London, Routledge.

Walther, G. (1984) 'Action proof vs illuminating examples?', *For the Learning of Mathematics* **4**(3), pp. 10-12.

Watson, A. (1994) 'My classroom', in Bloomfield, A. and Harries, T. Teaching, *Learning and Mathematics,* Derby, Association of Teachers of Mathematics.

Watson, A. (1995) 'Generalising', personal communication.

Watson, R. (1994) 'Runs of composite integers and the Chinese Remainder Theorem', *Mathematical Gazette,* **78**(482), pp. 167-172.

Weiner, B. (1972) 'Attribution theory, achievement motivation, and the educational process', *Review of Educational Research,* **42**, pp. 203-215.

Wells, C. G. (1979) 'Learning and using the auxiliary verb in English', in Lee, V. (ed.), *Language Development*, London, Croom Helm, pp. 250-270,

Westcott, M. R. (1968) *Towards a Contemporary Psychology of Intuition,* New York, Holt, Rinehart and Winston.

Wheeler, D. (1984) 'Gatherings', *Mathematics Teaching,* **106**, pp. 24-5.

Whewell, W. (1858) *Novum Organon renovatum: being the second part of The Philosophy of the Inductive Sciences*, London, John W. Parker.

Wierzbicka, A. (1976) 'Particles and Linguistic Relativity', *International Review of Slavic Linguistics* **1**(2-3), pp. 327-367.

Wilder, R. (1965) *The Foundations of Mathematics,* (second edition), New York, John Wiley.

Williams, R. A. J. (1995) 'Mathematical Conversations', Unpublished essay, Cambridge, Homerton College.

Williamson, T. (1994) *Vagueness*, London, Routledge.

Wills, D. D. (1977) 'Participant deixis in English and baby talk', in Snow, C. and Ferguson, C. (eds), *Talking to children*, Cambridge, Cambridge University Press, pp. 271-308.

Wood, D. J., Bruner, J. S. and Ross, G. (1976) 'The role of tutoring in problem solving', *Journal of Child Psychology and Psychiatry*, **17**(2), 89-100.

Wallwork, J. F. (1969) *Language and Linguistics*, London, Heinemann.

Woods, P. (1980) 'The development of pupil strategies', in Woods, P. (ed.), *Pupil Strategies: explorations in the sociology of the school*, London, Croom Helm, pp. 11-28.

Zadeh, L. (1965) 'Fuzzy Sets', *Information and Control*, **8**, pp. 338-53.

Zandvoort, R. W. (1965) *A Handbook of English Grammar*, third edition, London, Longmans.

Zaslavsky, C. (1973) *Africa Counts: number and pattern in African culture*, Boston, MA., Prindle, Weber and Schmidt.

APPENDICES

BLANK IN ORIGINAL

APPENDICES

BLANK IN ORIGINAL

APPENDIX 1: INDEX OF TRANSCRIPTS

SUSIE

Code	*Interviewer*	*Child*	*Age*	*Date (YearMonthDay)*
S1	Tim	Susie	9	910422
S2	Tim	Susie	9	910429
S3	Tim	Susie	9	910513
S4	Tim	Susie	9	910520
S5	Tim	Susie	10	920115

SIMON

Si0	Tim	Simon	11	9009
Si1	Tim	Simon	12	911226
Si2	Tim	Simon	12	911229
Si3	Tim	Simon	12	911231

BEADS

B1	Tim	Inciya/Sarah	11/11	920303

MAKE TEN

T1	Tim	Roksana/Anna	11/10	920612
T2	Tim	Lucy/Rachel	10/10	920617
T3	Tim	Alex/Caroline	10/11	920708
T4	Tim	Harry/Alan	10/11	920708
T5	Tim	Jubair/Shofiqur	11/10	920708
T6	Tim	Ishka/Frances	10/10	920716
T7	Tim	Anthony/Sam	11/11	920716
T8	Tim	Runa/Kerry	11/11	920716
T9	Tim	Rebecca/Runi	10/11	930316
T10	Tim	Susan/Shahnaz	11/11	939316

WHOLE SCHOOL AGE-RELATED STUDY OF 'MARKERS'

M1-45	Mark	Various	4:7 to 5:9	9406/07
M46-114	Mark	Various	5:10 to 7:9	9406/07
M115-180	Mark	Various	7:10 to 9:9	9406/07
M181-230	Mark	Various	9:10 to 11:9	9406/07

DONATED BY MEMBERS OF THE INFORMAL RESEARCH GROUP

IRG1A	Sue	Rebecca	4:8	9503
IRG1B	Sue	Jane	4:8	9503
IRG1C	Sue	Anna	4:3	9503
IRG1A	Sue	Jason	4:11	9503
IRG2	Ann	Charlie	10	9503
IRG3	Hazel	Faye/Donna	10/10	950317
IRG4	Kevin	Andrew/Matthew	10-11	9503
IRG5	Judith	Allan	13-14	9505
IRG6A	Rachel	Juliette/Di	18/18	950308
IRG6B	Rachel	Clare/John	18/18	950308

UNDERGRADUATE NUMBER THEORY PROJECT SUPERVISIONS

NT1	Tim	Lorna	21	950306
NT2	Tim	Caroline	21	950307
NT3	Tim	Nicola	21	950307
NT4	Tim	Jonathan	24	950313
NT5	Tim	Lorna	21	950313
NT6	Tim	Claire	21	950313
NT7	Tim	Katy	21	950314
NT8	Tim	Nicola	21	950314
NT9	Tim	Claire	21	950317
NT10	Tim	Caroline	21	950320

APPENDIX 2

DIVIDING BY ¾: SIMON AND SUSIE

Oh dear white children casual as birds,
Playing among the ruined languages,
So small beside their large confusing words.
[W. H. Auden, *Hymn to St. Cecilia*]

BLANK IN ORIGINAL

APPENDIX 2: DIVIDING BY ¾ - SIMON AND SUSIE

> He who seeks for methods without having a definite problem in mind seeks for the most part in vain. 1st November 1995
>
> (David Hilbert, *Bulletin of the American Mathematical Society 8*, p. 444)

I prepared the essay in this appendix in the course of thesis-writing; it seemed to spill off the pen as I concluded Chapter 4. I place it here because it addresses, not my eventual research question, but my original one: to access and describe children's cognitive frameworks and private mathematical constructions.

> Everyone - almost everyone - agrees that our goal is the study of the mathematical mind in action. (Ginsburg, 1981, p. 4)

Onslow (1991), writing about "real world" representations of abstract mathematical symbolism, analyses students' approaches to the problem 10 ÷ ½ presented in and out of context. He comments that this is

> A question which often provides some indications as to how a person has learned mathematics. (p. 33)

My interest was in how pupils bring and apply the framework of knowledge that they possess, in the solution of non-routine problems. The problem in this case can be represented as 100 ÷ ¾. I judged that the two children, Simon and Susie, would find a presentation such as "How many lots of three-quarters are there in a hundred" perfectly comprehensible but fairly demanding. I avoided any context (such as how many 75p bottles of coke you can buy for £100) partly because the context can be patronising, transparently spurious; but mainly because any context imposes part of the imagery that the student draws on to represent and solve the problem.

SIMON

The following vignette from 1990 predates my systematic collection of mathematical conversations. Simon, my son, was 11½ at the time.

S0:1	Tim:	How many three-quarters are there in a hundred?
2	Simon:	Well, there are seven three-quarters in ten, remainder a quarter.
3	Tim:	How many?
4	Simon:	No ... there are thirteen, remainder a quarter. Is that right?
5	Tim:	How did you get it?

6 Simon: Thirteen threes are thirty-nine, so there's a quarter left. [Pause] Thirteen times ten is a hundred and thirty. A quarter times ten is two and a half. Two and a halfs into three quarters goes three times remainder a quarter. So it's a hundred and thirty-three remainder a quarter.

7 Tim: [genuinely unsure] Are you sure that's right?

[Simon finds a calculator and keys in 100÷0.75. Gets 133.3333333 on the display.]

8 Simon: That means a hundred and thirty-three and a third.

9 Tim: Why a third when you said remainder a quarter?

10 Simon: Dunno. [goes away]

11 Tim: [3 minutes later] A hundred and thirty-three and a third lots of what?

12 Simon: Oh, the third is a third of three-quarters, so its a quarter. [looks quite pleased].

The problem is presented in terms of 'quotition' (that is, separation into an unknown number of parts, each of a given size; this is contrasted with 'partition', meaning separation into a given number of equal parts, of unknown size). Simon's approach is strategic, local, and uses his place-value knowledge, implementing the identity $ab \div x = a(b \div x)$. He decides that it will be easiest first to find how many three-quarters there are in 10. After an initial error is corrected in response to my prompt [S0:3], he arrives at 13, "remainder a quarter". He doesn't actually tell me *how* he 'got' it [5]; instead, he demonstrates [S0:6, to the pause] by multiplication *that* his answer is correct, recognising that 'how many ¾ are there in' and 'multiplying by ¾' are inverse functions. A plausible guess at how he (eventually) arrived at 13 is that he represented 10 as 40 quarters, then partitions 40 into 3 equal parts.

The calculator enables him to demonstrate that he knows the decimal equivalent of ¾, and that 'how many ¾ are there in' is formally modelled by division by ¾ . His mental calculation [6] of 13 remainder ¼ (in effect modulo ¾) multiplied by10 is fluent - it leaves me standing, in fact [7]. With my prompt [11] he reconciles (makes sense of) the calculator display with his mental calculation. His cognitive style is structural, algebraic; the structure holds the problem together whilst he deals with the component parts of it, finally synthesising a solution.

SUSIE

At my fifth and last transcribed interview [Appendix S5] with Susie, I posed the same problem. She was just 10 at that time. I judged that Susie, like Simon, would find the problem non-trivial but accessible.

I referred to my last *transcribed* interview with Susie: in fact I had interviewed her just before the Christmas holiday, and come away very excited about her way of dividing by fractions. Unfortunately the tape recorder had failed. I audio-taped a memo recording, as faithfully as I could, the course of events in the interview. I returned a month later, levelled with Susie about what had happened, and explained that I would ask her some of the same questions again, but that she should not go out of her way to repeat what she had said previously. I was aware, however, of her excellent recall of our conversations from week to week and that she would be likely to draw on what she remembered. It really didn't matter either way. As before, I began the interview with questions intended to probe her perception of fractions.

S5:1	Tim:	Right, I want you, to start with Susie, to explain to me what you mean by three-quarters.
2	Susie:	Well, if you have one thing, whole thing, and you cut it in half, and then the two bits in half again, and take away three of them, and take away one of them, the three left are three-quarters.
3	Tim:	What about five-sevenths?
4	Susie:	Well once again, if you cut a cake like that, [draws] cut it into seven equal pieces, it has to be equal ...
5	Tim:	has to be equal pieces ...
	[Here Susie comments on the fact that this is more difficult to draw]	
10	Susie:	It has to be equal, so if you take away two of those pieces you've got five left.

I shall return to the significance of this in a moment. Following the preamble, I quickly got to the point:

S5:11	Tim:	Right. Right. Now the next thing I want you to think about is, how many lots of three-quarters are there in a hundred?
12	Susie:	How many lots of three-quarters are there in a hundred.
13	Tim:	Yes.

14 Susie: Well ... oh I begin to remember what it was I did.

15 Tim: OK.

16 Susie: Em, I think what I did, it, wasn't it ... a take away or add sum ... how many lots of three-quarters ... Yes, I remember, I remember. I think what I did is have one hundred, add ... what was it? ... oohhh ... It's impossible to third it - a hundred, you need a third of a hundred. So that must be [writes] three ... thirty-three point three recurring [writes 33.3r, changes r to R] I'll put a capital R for that because that [r] means remainder. Ah, so if you add those two together, together, it should be one hundred and thirty-three point three recurring.

In [14] Susie presumably refers to the fact that I had asked her the same question a month ago. Susie's account [16] agreed with my recollection of the method she had used a month earlier. To find how many ¾ there are in 100, she adds to 100 a third of 100. My attempt to find out why she believes (with good cause) that it works does not meet with success.

17 Tim: Right, so it's a hundred and thirty-three point three recurring. Now what I actually asked you, right, was how many lots of three quarters in a hundred. And what you've done is to take a hundred, and then a third of a hundred, and add it on. Now, can you explain to me how that tells you how many three-quarters there are in a hundred?

18 Susie: Well, it doesn't tell me. I actually have worked that out quite a long time ago; worked that out, I did. It hasn't ... I did actually do some maths and worked it out that way [inaudible]. I did try doing it that way, and then tried doing it another way. It worked, but the other way I had was too difficult, so I just stuck to this way.

19 Tim: When did you work out this other way?

20 Susie: I can't remember ... well ...

21 Tim: You can't remember.

22 Susie: No.

In fact I had had no more success before Christmas. As an example of probing, contingent questioning for the purpose of cognitive research, it is no paradigm. I am still unclear about much of what she is saying in [18]. What exactly did she work out "quite a long time ago"?. Is she referring to our conversation last month? What was this other way? Invert and multiply? It seems that she had not been taught that method at school. In any case, it had been clear in December that Susie was devising and adapting her current method as we spoke, and (having found how many ¾ in 100) it had been her spontaneous but provisional suggestion that she could try something similar with four-fifths. This time, when I gave the lead, she was less tentative.

23	Tim:	OK. What about, how many four-fifths in a hundred?
24	Susie:	Four-fifths. It's correct, isn't it? [refers to 133.3R written]
25	Tim:	Oh yes, that's correct.
26	Susie:	Em, that was three ... what was that again? That was three-quarters.
27	Tim:	That was three-quarters in a hundred, now we're talking about four-fifths.
28	Susie:	That must be a fourth, this must be a fourth of it ... mmm ... a fourth of a hundred [pause]
29	Tim:	A fourth of a hundred. OK, so what have you got?
30	Susie:	One hundred and twenty-five.
31	Tim:	OK. Describe what you do again. Just describe.
32	Susie:	I just had the hundred, and then I had one fourth of hundred, and added them together.

Before long Susie has explained that the method won't work with seven-ninths because "it has to be one different" [S5:36]. Her division of 100 by 6 (for six-sevenths) is a virtuoso performance [Appendix S5:44-63]. In effect, she writes that half of 33.333... is 15+1.5+.15+.015+ ... (half of each 3); adding the units, tenths, hundredths and so on she gives 16.6666... "And it will continue happening. Six and carry five, six and carry five ...". So there it is: recurring decimals are convergent infinite series.

Next, I challenge her to expand the schema. No problem:

S5:62	Tim:	OK. Now, you said that it wouldn't work for seven-ninths didn't you, this method. Right? Now, I'd just like you to write down five-sevenths, just here.
63	Susie:	I'm going to have to think though, very well. Um, I'll try ... [pause]. Ahh, of course ... [interrupted]
64	Tim:	You have a think while I push the door up.
65	Susie:	... you can't ... I don't understand. It's definitely a hundred. So that means two ... Ahh, ahhh [*big moment*] you've got two left, and you need five each time. So if you have two hundred ... um ... divided by five. How many times does five go into two hundred? Well, it goes into one hundred twenty times ...

66 Tim: Mm-hm

67 Susie: Must go into forty times. So that's ... a hundred and forty. [...]

69 Susie: That's a two difference.

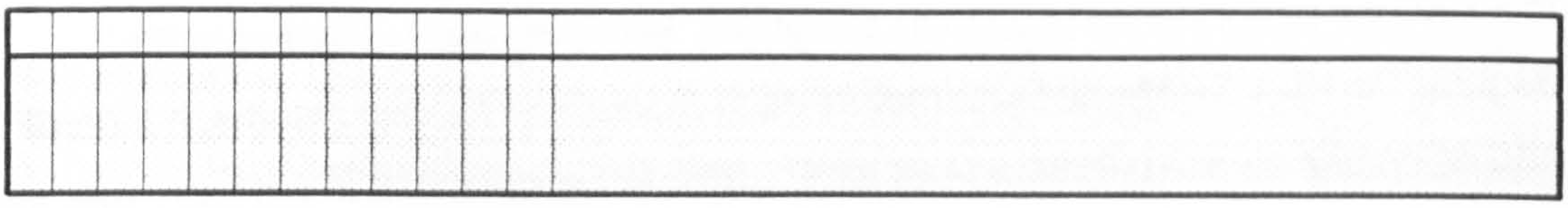
She articulates and extends the generalised process (by now consistently deploying 'you' as vague generaliser):

89 Susie: The next one, when you have two difference, you have to do two hundred divided by that, the number, and add a hundred to the equals.

90 Tim: Right, add a hundred to the equals.

91 Susie: And with three you have to do three hundred divided by whatever is the top number.

How does Susie conjure up these algorithms, progressively adapted for increasingly general application? I did have a hypothesis in the interview, but failed to elicit a direct, confirming account from Susie by contingent questioning. I believe, however, that there is indirect evidence, consistent with my hypothesis, as follows. I surmise that Susie takes a global view of the division problem, in which she first imagines *one* three-quarter part of each of the 100 'things'. I don't know what her things are. My corresponding image would be a long thin rectangular strip, standing on a 'base' 100 units long, with ¾ of the height blocked out, or shaded.



Thus, 100 "lots of" ¾ accounts for ¾ of the whole. The remaining ¼ is one third of the shaded part. Addition of one third of the part already accounted for will complete the whole. The essence of the imagery is the perception of the fraction (¾) in relation to the complement (¼) which is one-third its size.

Pirie, Martin and Kieran (1994) asked three categories of students (school and university) a set of six (written) questions about fractions, including: How would you explain ¾? Pirie and her collaborators describe four "major images" which emerged from the questionnaires:

1. Division: a quantity divided by a quantity.

2. Part of a whole.

3. A number of some sort.

4. A way of writing: a number over another number.

I suggest that Susie's image conforms to none of these stereotypes, being crucially a *part-complement* image. I believe that this perception was conveyed and reinforced by Susie in three recorded utterances:

(i) in the 'rulers' episode described earlier. Describing a tenth:

> If you have ten, and you take away **nine** ones, you have just the **one** left ... it's because you take away a ninth ... no, nine-tenths. So there's one-tenth left.

For Susie, a tenth is the remnant when nine-tenths is taken from the whole.

(ii) during the interview S5 itself, when at the beginning she described three quarters in the clearest part-complement terms.

S5:2 Susie: Well, if you have one thing, whole thing, and you cut it [in four] and take away **one** of them, the **three** left are three-quarters.

(iii) again in S5 she is consistent in conveying this part-complement representation in drawings and in her choice of words.

S5:3 Tim: What about five-sevenths?

10 Susie: It has to be equal, so if you take away **two** of those pieces you've got **five** left.

Each fraction is constructed by removing its complement from the whole. Evidence presented by Maher *et al.* (1994 can be interpreted to suggest that other pupils make a similar construction of fractions. Fourth grade children were asked to place some fractions on a number line. One child, Alan, wrote one-third at both the one-third and the two-thirds positions on the [0,1] interval.

To conclude interview S5, I challenged Susie to adapt her method to divide by improper fractions, such as five-thirds. It was just an idea, speculative. She rose to the challenge heroically [Appendix S5:158-190], this time *subtracting* a fifth of 200 from 100. Presumably she perceives five thirds as having a complement of negative-two thirds within the whole, and hence the need to trim away the excess two-fifths (from 100 lots of five thirds). She kindly enquires of me

187 Susie: ... Do you understand that?

To be honest, my answer, "Yes thank you" was a terse response to being patronised! Yet her apparent sense of superiority was not out of place. My understanding was operational, instrumental, whereas I knew that hers was relational.

SUMMARY.

Both Simon and Susie achieve novel and effective solutions to the quotition problem with fractional parts. Their cognitive styles and approaches are radically different, however.

Simon's solution exploits his secure sense and command of the structure of the elements of the problem and he confidently navigates his way through it. At any time he needs only to attend to a 'local' problem; the algebra will keep the rest 'on hold' and ensure successful synthesis of the solution.

Susie brings unusual but powerful fraction imagery to create a global, gestalt overview of the problem and a highly idiosyncratic solution. She quickly perceives the possibility of reconstructive generalisation (Harel and Tall, 1991) and she is able to adapt her method to a solve a comprehensive class of related problems.

Whilst the data from these particular interviews are rich in the cognitive contrasts they offer, such cognitive insights and conjectures were commonplace during my mathematical conversations with children and students.

APPENDIX 3

SELECTED INTERVIEW TRANSCRIPTS

Si1	Simon	26th December 1991
S5	Susie	15th January 1992
T6	Ishka and Frances	16th July 1992
IRG5	Allan	May 1995
NT4	Jonathan	13th March 1995

BLANK IN ORIGINAL

TRANSCRIPT Si1: SIMON 26th December 1991

Simon, my son, was aged 12 years 9 months.

1 Tim: I'd like you to give me two numbers that add up to ten.

2 Simon: Four and six

3 Tim: Two other numbers?

4 Simon: Five and five

5 Tim: Any others?

6 Simon: Three and seven, two and eight, one and nine. That's it. Well, there's nought point five and nine point five, and eight point five and one point five.

7 Tim: Right, you've gone from whole numbers to decimals.

8 Simon: Yes.

9 Tim: OK. What other sorts of decimal answers could there be?

10 Simon: Em, what harder, like more decimal places?

11 Tim: Whatever you think?

12 Simon: Em, nought point two five and nine point seven five.

13 Tim: Right.

14 Simon: And five point five and four point five, like that.

15 Tim: What if I gave you one of the numbers, one point three recurring, what would the other number be?

16 Simon: Em, eight point six recurring.

17 Tim: Why?

18 Simon: Because one point three recurring is basically a third ...

19 Tim: You mean the point three ...

20 Simon: ... point three recurring is basically a third, so you need ... well, the one, that's one, so to make it up to nine you add on eight, then you need another two thirds, which is point six recurring.

21 Tim: If you have, um, point three and point six recurring, and you add them up, what do you get?

22 Simon: Point nine recurring. Mmm - nearly one.

23 Tim: Nearly one.

24 Simon: Yes.

25 Tim: Why nearly one?

26 Simon: Because it's not, because point three isn't, it's just nearly a third. It doesn't quite get to the third.

27 Tim: When it's point three recurring.

28 Simon: Yeh.

29 Tim: Oh, so point three recurring isn't really a third at all?

30 Simon: Well. it's very nearly a third.

31 Tim: Very nearly a third.

32 Simon: Yeh.

33 Tim: What about point six recurring?

34 Simon: Very nearly two thirds.

35 Tim: So really, when you add them up, you don't get ten.

36 Simon: You just get nearly ten.

37 Tim: How nearly?

38 Simon: Um, point six and point three added together, that close.

39 Tim: (laughs) But you don't get exactly ...

40 Simon: ... exactly one.

41 Tim: OK, you've given me so far decimal numbers that add up to ten, and whole numbers that add up to ten. Are there any other sorts of answers that you could give me?

42 Simon: Minus numbers.

43 Tim: Can you give me an example of that? Remember that you've got two numbers that add up to ten.

44 Simon: Add up to ten. Minus five and minus eight. No, they don't. No. Minus five and minus eight?

45 Tim: Minus five and minus eight add up to ten?

46 Simon: Yes.

47 Tim: Can you explain that to me?

48 Simon: No they don't. Don't ...

49 Tim: Why not?

50 Simon: Dunno. I did them two weeks ago, but I've forgotten.

51 Tim: Well, if you have minus ...

52 Simon: Minus two and minus eight is minus ten.

53 Tim: OK, but suppose you want them to add up to ten.

54 Simon: Dunno.

55 Tim: Would it help to write it down?

56 Simon: Think so. No, I've forgotten.

57 Tim: Write down the whole numbers you know ....

58 Simon: Ah, ah, two add minus twe.... add twelve.

59 Tim: Minus two add twelve.

60 Simon: Yes.

61 Tim: OK, why did you suddenly think of that?

62 Simon: Um, because adding is going up towards north and then onwards. So minus two is two away from nought, and then ten up to ten.

63 Tim: OK, so what if you had minus five as one of your numbers?

64 Simon: Fifteen. Fifteen add minus five.

65 Tim: OK. So you've got decimal numbers, some negative and positive numbers, whole numbers, positive numbers. Anything else?

66 Simon: Fractions?

67 Tim: Fractions. Can you give me an example?

68 Simon: Um, seven and one fifth and two and four fifths.

69 Tim: Right [pause]. Suppose you just take the whole number ones. Can you ... you gave me for example four and six, and then five and five, at the beginning, and now we're only taking positive numbers. How many different solutions are there?

70 Simon: Five ... no, six.

71 Tim: Six. OK. Are you counting two and eight and eight and two as being the same or as different?

72 Simon: The same.

73 Tim: The same. OK.

74 Simon: Otherwise there's twelve.[pause] No, eleven.

75 Tim: Why did you change your mind?

76 Simon: Because five and five is the same as five and five.

77 Tim: OK. Suppose you count two and eight and eight and two as different; then you're saying there's eleven ways?

78 Simon: Yes.

79 Tim: If instead of numbers adding up to ten ... supposing I'd said numbers adding up to fifteen ...

80 Simon: Two numbers adding up to fifteen?

81 Tim: Two numbers adding up to fifteen, whole positive numbers. How many answers would there be to that?

82 Simon: Um [pause while he writes]. Sixteen.

83 Tim: Sixteen ways. What have you written there?

84 Simon: One to fourteen, two thirteen, three twelve, four eleven, five ten, six nine, seven eight, eight seven. So that's the boundary. So those numbers and then nought and fifteen.

85 Tim: Oh, you're counting nought and fifteen?

86 Simon: Yes, otherwise there's fourteen ways. And fifteen with nought.

87 Tim: Wait a minute, when you said that for ten there were eleven ways ...

88 Simon: That was including nought and fifteen and fifteen and nought. No, nought and ten and ten and nought.

89 Tim: I mean, are you going to count that or not?

90 Simon: Yes.

91 Tim: OK. So there's sixteen ways for fifteen if you count nought and fifteen.

92 Simon: And fourteen ways if you don't.

93 Tim: Right, shall we decide whether we're going to count nought and the number or not.

94 Simon: You decide.

95 Tim: You want me to decide.

96 Simon: I think we should.

97 Tim: You think we should. Alright then, we'll include it. It doesn't matter to me. Um, suppose instead of fifteen I said sixteen. How many ways would there be?

98 Simon: Um, seventeen, I think?

99 Tim: Why do you think that?

100 Simon: Because the other ones, for ten and fifteen, there's been one more.

101 Tim: There's been one more.

102 Simon: Yes.

103 Tim: So for a hundred there would be how many ...

104 Simon: A hundred and one.

105 Tim: A hundred and one ways, right. Now that worked for the others, are you sure it will work for these other ... say sixteen or a hundred or whatever.

106 Simon: [sneezing fit] Um, no.

107 Tim: Would you like to think about a reason as to why it might work?

108 Simon: [pause] No, can't ...

109 Tim: Well, could you have a go at talking me through what it would be like, if you were to work out all the ways for twenty, and see whether that suggests why it is in fact twentyone. You can write them all down if you want, but it might be better if you didn't.

110 Simon: Yeh [writes 0-20, like that]. Twenty, it's like going up a scale from nought to twenty. Like the left hand *number goes from nought to twenty, and there* are twentyone numbers inbetween nought and twenty.

111 Tim: Inbetween.

112 Simon: No, including.

113 Tim: Right.

114 Simon: So you've got twentyone possibilities, cos the other [inaudible]'s going in the other direction.

115 Tim: Right, right. And what if I'd said adding up to a hundred?

116 Simon: The same. There are, between nought and a hundred, there are a hundred

and one ... well, including a hundred, and that goes in both directions.

117 Tim: So what's the rule, with any number?

118 Simon: It's that number, plus one.

119 Tim: Right, and you're sure about that.

120 Simon: Yeh.

121 Tim: OK.

122 Simon: Or, if you don't include nought, minus one.

123 Tim: Oh, it's one less if you don't include the number? Why one less?

124 Simon: Because you're getting the numbers between nought and that number, and its always n [*enn*] minus one. If you have, if seventeen's your number, then you do, you go, the numbers between nought and seventeen, there are sixteen of them, and then it's one to sixteen and then one to sixteen, going down and up. So there are only sixteen numbers.

125 Tim: Yeah. I suppose it's a bit, is it a bit funny counting like twenty and nought and nought and twenty. I suppose it's no different from counting one and nineteen and nineteen and one, is it.

126 Simon: No, I don't think so.

127 Tim: Right. Let's um, try now, um, this problem.

128 Simon: Right.

129 Tim: The basic problem is, I'd like you to give me three numbers that add up to twenty.

130 Simon: One, two and seventeen.[writes]

131 Tim: OK. And again we're allowing different orders, OK?

132 Simon: What do you mean?

133 Tim: Well, like seventeen, two and one, they'd all be different.

134 Simon: Yeah.

135 Tim: Yeah?

136 Simon: Yeah.

137 Tim: Right, now give me another one.

138 Simon: One, three and sixteen. [writes]

139	Tim:	Right. Another one?
140	Simon:	One, four and fifteen. [writes]
141	Tim:	OK. You've written one, fourteen and fifteen. [Simon corrects] You want three numbers that add up to twenty. Right?
142	Simon:	Mm.
143	Tim:	Do you think you can work out how many different ways there are of doing that?
144	Simon:	Um ...
145	Tim:	Do you want a bit of time to think about that?
146	Simon:	Yeah, please. [pause] Can I have one number included twice?
147	Tim:	Yes.
148	Simon:	Is like one, eighteen, ones different from one, one, eighteen?
149	Tim:	Say that again, please.
150	Simon:	One ...
151	Tim:	Oh, one, eighteen, one ...
152	Simon:	different from one, one, eighteen?
153	Tim:	Um ... well, let's say it is, yes.
154	Simon:	Right. [pause. Simon writes 5-14, 6-13, ..., 9-10, 10-9]
155	Tim:	Explain to me what you're doing at the moment.
156	Simon:	Um, I'm just working this out because if I have one at the beginning, then I'm working out the number of ways that you can have for the other two numbers.
157	Tim:	OK, and how many is that?
158	Simon:	Eighteen. Or, of course, there's nought. One, nought, nineteen, which makes it so there are twenty different ways of doing it.
159	Tim:	OK. [pause] As we're trying to make it clear that there are three numbers here, I think if we have one of them nought it might just complicate things. So I suggest that we don't include nought on this occasion.
160	Simon:	Alright, OK. So I've got eighteen possibilities for that. So I've got eighteen for one [inaudible] and eighteen for the other. I wouldn't include twenty would I, as one of my first numbers, cos I can't have twenty, nought,

nought.

161 Tim: Right, OK. If you start with one then you've got eighteen possibilities.

162 Simon: So two's going to have either six.. sixteen or seventeen. I'll check that. [writes 2-1-17, 2-2-16, 3 15, 4 14, ... , 9 9] One seventeen, two sixteen, ... [mumbles]. One, two, three, four, five, six, seven, eight, ...[counts under his breath] ... sixteen, seventeen ways instead of eighteen. So if two has seventeen, three's going to have sixteen. So that's just like that, isn't it. So if one has eighteen, then two's going to have seventeen, and three sixteen and four fifteen. So shall I work that out?

163 Tim: Mm-hm.

164 Simon: Um, it's just one add two add three add four add five addsix add seven add eight add nine add ten add eleven, and twelve, thirteen, fourteen, fifteen, sixteen and seventeen. And eighteen.

165 Tim: And eighteen [inaudible]

166 Simon: Yeh

167 Tim: Right. Um ...

168 Simon: There's triangular number eighteen.

169 Tim: Yeh ... pardon?

170 Simon: There's tri... eighteenth triangular number.

171 Tim: [laughs] Eighteenth triangular number, OK. You said it's one add two add three and so on. Where would you get one? I mean, you said that if you start with one you get eighteen ways; when would you get one way?

172 Simon: Um, good point. Eighteen doesn't count then, cos that's going to have to be nineteen and one and nought.

173 Tim: [pause] No, nineteen wouldn't ...

174 Simon: count ...

175 Tim: count ...

176 Simon: as the first number.

177 Tim: Mm-hm.

178 Simon: So it's eighteen is one and one, so that eighteen has one ...

179 Tim: Right, OK.

180 Simon: and seventeen has two. So I'm just going to write this out.

181 Tim: OK.

[pause whilst Simon writes 18-1, 17-2, ..., 9-10]

182 Simon: Yeh, so it goes from one to twenties (??), so it's the triangular numbers from one to eighteen.

183 Tim: Right. Can you work out the eighteenth triangular number?

184 Simon: Yeh. It will take a bit of time.

185 Tim: How would you do it?

186 Simon: Add one and two and three and four and so on.

187 Tim: OK. Suppose instead of doing all that ... it will take a bit of time, OK? ...

188 Simon: Yeh, I'm sure there's a quicker way of doing it.

189 Tim: Well, imagine all the numbers from one to eighteen, right?

190 Simon: Yeh

191 Tim: Now add the first and the last.

192 Simon: Eighteen and one.

193 Tim: Yeh

194 Simon: Nineteen.

195 Tim: Now add the second and the ... penultimate.

196 Simon: Two and eighteen, which is twenty.

197 Tim: No, you're adding the first and the last ...

198 Simon: Yeh

199 Tim: and then the second and the ...

200 Simon: Oh. two and seventeen ... is nineteen. So it's nine nineteens.

201 Tim: OK.

202 Simon: Nine nineteens [writes in column format] are a hundred and seventy one. Is that right?

203 Tim: Right, yeh, good. Brilliant.

204 Simon: So there are a hundred and seventyone different ways of adding like the numbers between one and nineteen together, one and eighteen together, to make twenty.

205 Tim: Right. Suppose instead of twenty, right, I said how many different ways are there of adding up three numbers to make fifty.

206 Simon: I'd do fortynine times [pause] twentyfive.

207 Tim: [pause] You'd better explain that.

208 Simon: Um, no, I wouldn't. I'd do fortynine times twentyfour.

209 Tim: Explain it.

210 Simon: Well, it's the triangular number of, em, it's the ... working out the triangular number of ... the fortyeighth triangular number.

211 Tim: Mm-hm. Why do you know that?

212 Simon: I'm going on the assumption that it works the same for twenty.

213 Tim: What happens with twenty?

214 Simon: It, em , I found the triangular number for eighteen, because ... the second number before twenty.

215 Tim: Right, right. So what you do with fifty, you say ...

216 Simon: Make, work out the triangular number fortyeight.

217 Tim: Right.

218 Simon: And to do that, I times it by ... so I do fortyeight times ... no, I do fortynine times half of fortyeight, which is twentyfour.

219 Tim: Right. Can you see why it's fortynine times half of fortyeight?

220 Simon: Yeh

221 Tim: Why?

222 Simon: Because, to work out a triangular number, you get the first and the last, and the second and that ...

223 Tim: and multiply it by how much?

224 Simon: Um, the num... a half of the number ... of ... half the number of numbers you've got. So it's like from nought to fortyeight, so half of that, cos you've only got half the numbers to work out.

225 Tim: Right. What if, um - that's good - I mean [afterthought] do you want to work out fortynine twentyfours?

226 Simon: No.

227 Tim: No? OK. Um, ...

228 Simon: Yeh, I will.

[Simon draws up a 2x2 grid with 2 4 on top and 4 9 down the l.h. side]

229 Tim: That's a whacky way of doing it.

230 Simon: To do that, I need to put two noughts in that way [writes 800]. Two's are eight, fours are sixteen [enters in grid]

231 Tim: Who showed you that way?

232 Simon: You did.

233 Tim: Did I? [laughs, Simon laughs] Damned good way! [fits of laughter]

234 Simon: I need to put in the nought here, don't I?

235 Tim: Mm-hm. Careful now, that's forty, right, so forty fours is ...

236 Simon: Oh, so forty times four is a hundred and sixty.

237 Tim: Mm-hm

238 Simon: So nine twos is a hundred and eighty.

239 Tim: Mm-hm

240 Simon: Four nines are thirtysix. So I've got eight hundred [writing in column for addition] one sixty, one eighty, thirtysix. Add six, six, six and eight's fourteen, seventeen, and eight, nine ten eleven. One thousand one hundred and seventysix.

241 Tim: OK, very good. You know, you said when you do, like the fortyeighth triangular number, you ... well, you explained what you did, OK?

242 Tim: Supposing you wanted to work out the, em, the fifteenth triangular number, right? Could you apply the same method to that?

243 Simon: Not totally.

244 Tim: Explain why.

245 Simon: Because you'd get to, um, seven and ... yes, you could.

246 Tim: Why, explain what you're thinking.

247 Simon: I was thinking that you'd get to seven and a half and seven and a half, and it wouldn't work. But it would.

248 Tim: Why?

249 Simon: No, no ...

250 Tim: What do you do?

251 Simon: I was thinking that you go one and fifteen, and two and fourteen, and ...

252 Tim: so you're going for sixteen all the time

253 Simon: Yeh, and three and thirteen,

254 Tim: yes

255 Simon: two and ... four and twelve, five and ten - no, five and eleven

256 Tim: Mm-hm

257 Simon: six and ten, seven and nine

258 Tim: Yeh.

259 Simon: Eight and, you're stuck there.

260 Tim: Yeh

261 Simon: So it's eight and a half ... seven and a half and seven and a half. You'd get stuck with one of the sixteens. You get an eight left over.

262 Tim: With fifteen you get an eight left over in the middle.

263 Simon: Yeh

264 Tim: Yeh

265 Simon: So ...

266 Tim: So what would you do?

267 Simon: You do, em, half of fifteen which is seven and a half. So you do, you saw (??) those two numbers, you saw the two whole numbers [inaudible] below. Seven you times by sixteen.

268 Tim: Right

269 Simon: and then you'd add the eight on the end.

270 Tim: You do seven sixteens and add eight.

271 Simon: Yeh

272 Tim: Yeh

273 Simon: That right?

274 Tim: Yeh, well it sounds ... yes, yes.

275 Simon: Seven sixteens are ... [writes in column format] um [mumbles] seven, six

sevens are fortytwo. Hundred and twelve add eight is hundred and ten. Hundred and twenty [laughs].

276 Tim: Right, so it gives you a hundred and twenty.

277 Simon: Yeh

278 Tim: Now, the ... the precise rule that you had before that worked, em, ...

279 Simon: That's for even numbers.

280 Tim: That's for even numbers.

281 Simon: Yeh.

282 Tim: and that was, you take half of the number, yeh?

283 Simon: Yeh

284 Tim: so in this case that would give you seven and a half.

285 Simon: Yeh

286 Tim: Yeh. Would that actually work in this case, with this odd number?

287 Simon: Not sure.

288 Tim: Well, work it out. What have you got to work out?

289 Simon: Seven point five times fifteen.

290 Tim: [pause] Fifteen?

291 Simon: No, sixteen.

292 Tim: Right, [inaudible] work that out. Do it on there.

[Simon writes 16 7.5 in vertical format]

293 Simon: [mumbles] I'm not totally sure how you do point fives, but ... do one seven is seven ...

294 Tim: Do you want a calculator?

295 Simon: Yeh [pause whilst one is found]

296 Tim: Right, you're doing sixteen times seven point five.

297 Simon: Yeh, sixteen times seven point five ... yeh, that works as well, anyway.

298 Tim: So it [rule] works, even when it [number] is an odd number.

299 Simon: Yeh

300 Tim: By the way, the seven point five, you could just call seven and a half,

couldn't you?

301 Simon: Yeh

302 Tim: Right

303 Simon: Yeh

304 Tim: What's seven and a half sixteens?

305 Simon: Oh, sixteen times seven, add eight.

306 Tim: Mm-hm [Simon laughs]. So it's [*answer*] the same. Right.

END OF SESSION

TRANSCRIPT S5: SUSIE 15th January 1992

Susie's 10th birthday was 24.12.91. I had interviewed her on Friday 13th December 1991, but unfortunately the tape recorder had failed. I returned a month later, explained what had happened to Susie, and that I would ask her some of the same questions again, but that she should not consciously attempt to recall what she had said, but to say what she thought now.

1 Tim:		Right, I want you, to start with Susie, to explain to me what you mean by three-quarters.
2 Susie:		Well, if you have one thing, whole thing, and you cut it in half, and then the two bits in half again, and take away three of them, and take away one of them, the three left are three-quarters.
3 Tim:		What about five sevenths?
4 Susie:		Well once again, if you cut a cake like that, [draws] cut it into seven equal pieces, it has to be equal ...
5 Tim:		has to be equal pieces ...
6 Susie:		Yes, it does, so that's hard [*to draw*], I can't do that, so I'll just do it as close as I can.
7 Tim:		Right.
8 Susie:		I can't [inaudible: "that's roughly seven"?]
9 Tim:		Right
10 Susie:		It has to be equal, so if you take away two of those pieces you've got five left.
11 Tim:		Right. Right. Now the next thing I want you to think about is, how many lots of three-quarters are there in a hundred?
12 Susie:		How many lots of three-quarters are there in a hundred.
13 Tim:		Yes.
14 Susie:		Well ... oh I begin to remember what it was I did [*reference to 13.12.91*]
15 Tim:		OK
16 Susie:		Em, I think what I did, it, wasn't it ... a take away or add sum ... how many lots of three-quarters ... Yes, I remember, I remember. I think what I did is have one hundred, add ... what was it? ... oohhh ... It's impossible to third it - a hundred, you need a third of a hundred. So that must be [writes] three ... thirty-three point three recurring [writes 33.3r, changes r to R] I'll put a capital R for that because that [*r*] means remainder. Ah, so if you add

those two together, together, it should be one hundred and thirty-three point three recurring.

17 Tim: Right, so its a hundred and thirty-three point three recurring. Now what I actually asked you, right, was how many lots of three quarters in a hundred. And what you've done is to take a hundred, and then a third of a hundred, and add it on. Now, can you explain to me how that tells you how many three-quarters there are in a hundred?

18 Susie: Well, it doesn't tell me. I actually have worked that out quite a long time ago; worked that out, I did. It hasn't ... I did actually do some maths and worked it out that way [inaudible]. I did try doing it that way, and then tried doing it another way. It worked, but the other way I had was too difficult, so I just stuck to this way.

19 Tim: When did you work out this other way?

20 Susie: I can't remember ... well ...

21 Tim: You can't remember.

22 Susie: No.

23 Tim: OK. What about, how many four-fifths in a hundred?

24 Susie: Four-fifths. It's correct, isn't it? [*refers to 133.3R written*]

25 Tim: Oh yes, that's correct.

26 Susie: Em, that was three ... what was that again? That was three-quarters.

27 Tim: That was three-quarters in a hundred, now we're talking about four-fifths.

28 Susie: That must be a fourth, this must be a fourth of it ...mmm... a fourth of a hundred [pause]

29 Tim: A fourth of a hundred. OK, so what have you got?

30 Susie: One hundred and twenty-five.

31 Tim: OK. Describe what you do again. Just describe.

32 Susie: I just had the hundred, and then I had one fourth of hundred, and added them together.

33 Tim: Right. What sort of ... can you give me another example of how many lots of something in a hundred, that you could do that way.

34 Susie: Well, you could, there, it's a fact you can, could not do the same way with ... seven...ninths [writes 7 9]

35 Tim: You couldn't do the same. Why not?

36 Susie: Because it has to be one different. Like with this one [*3/4*] it was one and that one [*4/5*] was one.

37 Tim: Has to be one difference. OK.

38 Susie: Yes, just one number, three, three-quarters [writes 3/4 and indicates 3,4].

39 Tim: One difference between which numbers?

40 Susie: The number of them and the quarter. And it has to be the smaller one to the number.

41 Tim: Right, [cough] could you, um, work out how many lots of six-sevenths there are in a hundred?

42 Susie: Ah [inaudible]

43 Tim: Six-sevenths, how many in a hundred?

44 Susie: Now what's a sixth of a hundred? Mm [pause], yes, I know a third of a hundred is that [indicates 33.3R]

45 Tim: Thirty-three point three ...

46 Susie: Yes, but it can't, um, a sixth will be half that.

47 Tim: Right, a sixth of a hundred will be half of that, yes.

48 Susie: So what's half of three. Ah, one ...

49 Tim: Can you work it out up here? [*away from written 33.3R*]

50 Susie: No, I'm working out this, so I can ... I will put it up there when I've worked it out. I need just to be close up there, which will help me work it out.

[Susie writes 15 beneath 33, crosses out 5 and replaces with 6, writes 5 beneath .3R - later changes that to 6 too]

51 Susie: So, do you know why I carried five into that? I had that one, and I had a half, so that I had that. Why I'm moving in this way, it's a bit different. In a normal adding sum you move the other way [*ie right to left rather than the left to right here*]

52 Tim: I'm not clear what you're doing Susie, you're working out a half of thirty-three point three recurring ...

53 Susie: So I worked out, three, I was half three, one fif... is one point five ...

54 Tim: That's half of three, yes.

55 Susie: So I just moved, put the five there. But now what I do is half that ... six ...

56 Tim: Er, wait a minute, where did the six come from? You had ...

57 Susie: I had five [*from 15*], and I have one of those [*from 1.5*], and then we carry the five ... over to there. And it will continue happening. Six and carry five, six and carry five, so it must be [pause] sixteen point six recurring.

58 Tim: Sixteen point six recurring. OK.

59 Susie: Which is one hundred and sixteen point six recurring.

60 Tim: And that was how many six-sevenths in a hundred, wasn't it?

61 Susie: Yes.

62 Tim: OK. Now, you said that it wouldn't work for seven-ninths didn't you, this method. Right? Now, I'd just like you to write down five-sevenths, just here.

63 Susie: I'm going to have to think though, very well. Um, I'll try ... [pause]. Ahh, of course ... [interrupted]

64 Tim: You have a think while I push the door up.

65 Susie: ... you can't ... I don't understand. It's definitely a hundred. So that means two ...Ahh, ahhh [*big moment*] you've got two left, and you need five each time. So if you have two hundred ... um ... divided by five. How many times does five go into two hundred? Well, it goes into one hundred twenty times ...

66 Tim: Mm-hm

67 Susie: Must go into forty times. So that's ... a hundred and forty.

68 Tim: One hundred and forty. OK. That's for five-sevenths.

69 Susie: That is a two difference.

70 Tim: Right, that's a two difference. So, can you show me how you'd do that, say, for five-eighths? How many lots of five-eighths in a hundred?

71 Susie: [mumbles, inaudible, then] six, seven, eight, so that's three difference.

72 Tim: Yeah. So what do you do?

73 Susie: Ah-ha. One hundred [writes 100], that's three hundred divided by five [writes 300÷5 below]. Which is sixty [writes 160]

74 Tim: So it's one hundred and sixty. OK. Can you explain ... you've obviously got a rule that depends on the difference. Right? Now supposing you've got some fraction, it's something over something else, right?

75 Susie: I don't know what you mean.

76 Tim: Right, well like five-eighths is five over eight.

77 Susie: Yes.

78 Tim: Now supposing you've got a fraction generally, that's some number on top of something else

79 Susie: Yes.

80 Tim: And I say to you, how many lots of that in a hundred? Explain to me what your method is.

81 Susie: Well, if you have one, one difference, you just take a hundred and add a third, or whatever it is on the next number above ... to the hundreds ... well, a hundred.

82 Tim: Right.

83 Susie: So you would have a hundred and - for three-quarters - a third of a hundred ...

84 Tim: Am-hm

85 Susie: ... and add them together will make the answer.

86 Tim: Am-hm

87 Susie: And so with anything, well, just take the top one, and third it, or eighth it, or whatever the top one, whatever the top number is, adding a hundred to it.

88 Tim: Right.

89 Susie: The next one, when you have two difference, you have to do two hundred divided by that, the number, and add a hundred to the equals.

90 Tim: Right, add a hundred to the equals.

91 Susie: And with three you have to do three hundred divided by whatever is the top number.

92 Tim: With three what?

93 Susie: Divided by whatever the top number is.

94 Tim: Yes, when you say "with three ..."

95 Susie: That's difference.

96 Tim: Oh, with a three difference. OK. And with a four difference?

97 Susie: It would be four hundred divided by the top number.

98 Tim: OK. [pause]

99 Susie: Add a hundred always.

100 Tim: Last time when we talked about this [*34 days earlier*] ...

101 Susie: Yes

102 Tim: ... you suddenly said something about four-fourths, as far as this rule was concerned.

103 Susie: Four-fourths. [pause] Must be a hundred.

104 Tim: How does your rule ...

105 Susie: It's very simple. When you get nought difference it means you add nothing to a hundred.

106 Tim: OK. Now here's a slightly strange one I want to ask. You know we've done three-quarters or five-eighths ...[interrupted]

107 Susie: Well we've actually, another thing that might work, if you did it with fifty, for instance, all that happens with one difference is, you do ... say three-quarters ..

108 Tim: Yes

109 Susie: all you do is have a third of fifty ...

110 Tim: Right

111 Susie: err ... add fifty. So you just have the number it is add, er, it's [inaudible] a hundred. [*not clear what she means here*]

112 Tim: Oh I ...

113 Susie: So with three it would be ...um... one hundred and fifty, three fifties you see, with three difference between the numbers. See?

114 Tim: No, say that one again.

115 Susie: Well, suppose you had ...um... I'll do five ... and we'll say eighths [writes], three difference.

116 Tim: Right.

117 Susie: How you would work that out, is fifty, three fifties because of the difference ..

118 Tim: Right, yes.

119 Susie: One hundred and fifty, and that divided by the top number, which is five.

120 Tim: Right, yes, yes, I get it, I get it. And that would be, that's how you do it with a hundred [points] and that's how you do it with fifty.

121 Susie: Yes

122 Tim: And with any number you do something like that?

123 Susie: Yes.

124 Tim: OK. Now - can I borrow your pencil? - so you've got a rule that works for five-eighths, OK

125 Susie: Yes, well any, any equal at all.

126 Tim: Sure, sure.

127 Susie: Any one at all.

128 Tim: Or three-quarters, or whatever.

129 Susie: Yes, but I have not got a rule for ones like ... it is a lot harder to have ones like ... [writes 4/3] that.

130 Tim: Ah, that's exactly what I was going to ask you! [Susie laughs]. You've written down four-thirds.

131 Susie: Yeah.

132 Tim: So the question is, how many lots of four-thirds are there, OK?

133 Susie: Yeah. In, in a ...

134 Tim: Well, let's say in a hundred, shall we?

135 Susie: Yeah, um, now ...

136 Tim: Can I just ...

137 Susie: Now, I think what you do is, now you do a takeaway. I think, I ... I don't know. So what you have is a hundred, take away a fourth, twenty-five, equals seventy-five. I think that probably will be the answer.

138 Tim: OK

139 Susie: I think so. Or it might be thirds. I'm not sure which it is. It would be possible to work it out in thirds wouldn't it?

140 Tim: Which thirds?

141 Susie: I'm not quite sure whether you either do it with the top number or the bottom number.

142 Tim: Which did you do it with before?

143 Susie: The top number.

144 Tim: So that would seem to be ...

145 Susie: Seventy-five [writes]

146 Tim: Right, right. Can we just check one thing? If I say "How many lots of, em, four-fifths are in a hundred?", and you get a hundred and twenty-five, alright? Would you expect the answer to be more than a hundred, if you're dividing by four-fifths?

147 Susie: I would have expected it to be less?

148 Tim: Why?

149 Susie: Because, this is more than one [indicates 4/3]

150 Tim: Ah, with this one you'd expect it to be?

151 Susie: Yes.

152 Tim: Is that right?

153 Susie: Yes.

154 Tim: So you're dividing by more than one, so you'd expect an answer less than a hundred? And you've got seventy-five

155 Susie: Yes

156 Tim: which is less than a hundred. So that's good.

157 Susie: That could be.

158 Tim: I've just had a thought, Susie, of something you could try. What about, you've done, you've tried your rule for four-thirds, right. Try it for two over one.

159 Susie: Two over one.

160 Tim: How many lots of two over one ...

161 Susie: That's easy, fifty.

162 Tim: Right.

163 Susie: Because the one is whole.

164 Tim: Right.

165 Susie: So it's just two, so it's half of a hundred. Yep, I've worked that out, just like that.

166 Tim: Good. Now what about this method that you're using here.

167 Susie: I don't know.

168 Tim: Well, what would you actually do?

169 Susie: I could work at these, two over one, I know the answer of that, see if it worked out like ... that way.

170 Tim: Well, try it this way and see if you get the right answer.

171 Susie: So that's a hundred [writes 100-50] take away - I know this must be fifty - so that must be correct.

172 Tim: That's correct. And why did you take away fifty?

173 Susie: Because that is a half of [points to 100]

174 Tim: Right. So that ...

175 Susie: That [points to 75], that I think is correct.

176 Tim: You now believe the seventy-five is correct. OK. [pause] What about, lets say now ... try this one. One hundred divided by five-thirds.

177 Susie: One hundred divided by five-thirds [sighs]. Right. See, this one you added it. I can't understand that. That one's harder. Ahh, Ahh, oh, yes, you would have twice as much for the difference, won't you. So that's ... so suppose we had five-thirds [writes 5/3]. Want five-thirds. You would have to take a hundred ...

178 Tim: Yes.

179 Susie: Take away ... um, a fifth of two hundred.

180 Tim: Right.

181 Susie: It would have to be two hundred, because of the amount in ...

182 Tim: Because of the ...

183 Susie: The space in here [points to 5/3].

184 Tim: The space of two.

185 Susie: Yeah. It shouldn't ... it's not really a space of two, it's a space of one. The space.

186 Tim: Right.

187 Susie: And it has to be two hundred. So you would have two hundred [writes] divided by five. Do you understand that?

188 Tim: Yes thank you.

189 Susie: And then you have two hun ... one hundred would be ... that's forty. That equals forty. So take away forty [writes] equals ... sixty.

190 Tim: Brilliant.

END OF SESSION

TRANSCRIPT T6: ISHKA and FRANCES 16 July 1992

Ishka is 10 years 6 months and Frances 10 years 3 months. Frances' family lives mostly in Japan. She is fluent in both English and Japanese. Frances had previously attended the school; on this occasion the family were "visiting" Cambridge for a month.

1	Tim:	OK. I'll explain to you what we're going to do. Erm, there's a pencil and some paper to write on if you want to but just for the moment I'd like you, Ishka, to tell me any two numbers that add up to ten.
2	Ishka:	Five and five [laughs]
3	Tim:	Five and five. Frances?
4	Frances:	Er [laughs] Nine add one.
5	Tim:	Nine add one. Some more?
6	Ishka:	Seven add three.
7	Frances:	Six and four.
8	Tim:	Right. Now what I want you to do ... you know we could go on doing that until we've used them all up ....
9	Frances:	Yeah.
10	Tim:	I'd like you two to agree between you ... incidentally we'll adjust that [microphone] Frances so it's not quite so close, right, um I'd like you just to - yours is fine Ishka - I'd like you two to agree between you, how many different ways there are of doing that. Right? Two numbers that add to ten, and I'll just be quiet for a moment.
11	Ishka:	Er ...
12	Frances:	There's one and nine.
13	Ishka:	Yeah.
14	Frances:	So that's one. Two and eight ... and then there's
15	F+Ishka:	Three and seven.
16	Frances:	Four and six, five and five, six and ... oh that's the same.
17	Ishka:	Five ways?
18	Frances:	Maybe.
19	Ishka:	Mm, maybe ... I think ....
20	Frances:	What do you think?

21 Ishka: We haven't had five five have we?

22 Frances: We have!

23 Ishka: Oh OK, erm ...

24 Frances: The others are like if you do six four, we've already done four six.

25 Ishka: Mm [sighs]

26 Frances: Shall we just say five ways?

27 Ishka: There's about five.

28 Tim: Erm, I'd like you to be more convinced Ishka. I mean if it's about five then it's four or six or seven or whatever ... the number's sufficiently small that I think you should be sure one way or another.

29 Frances: I think it's five ways.

30 Ishka: But I'm sure.

31 Tim: You are sure.

32 Frances: Me too.

33 Tim: OK. Don't play with the [microphone] wire, OK. Erm, right so for ten there's five ways ... incidentally you discussed or mentioned as to whether four and six, and six and four were going to be different, yup? ... and are you, you've decided, have you, that you're going to count them as the same?

34 Ishka: Yeh.

35 Frances: Yeh.

36 Tim: Yeh? OK. Right, I should now like you to consider the same problem, only this time we're going to consider the number of different ways of making twenty. Right, so, I mean let's just get started. Ishka, give us, give me any two numbers which add up to twenty.

37 Ishka: Fifteen and five.

38 Tim: OK ... and Frances?

39 Frances: Ten and ten.

40 Tim: So you understand the problem, and again I'd like you to agree how many ways are there to do that.

41 Frances: So ... there's one and nineteen.

42 Ishka: There's ....

43 Frances: ....two and eighteen, three and

44 F+Ishka: seventeen

45 Frances: Four and sixteen ...

46 Ishka: Yup.

47 Frances: Five and fifteen, six and ... six and fourteen.

48 Ishka: Yup.

49 Frances: Seven and thirteen.

50 Ishka: Mm.

51 Frances: Eight and twelve.

52 Ishka: Yeah.

53 Frances: Nine and eleven, ten and ten, so that's ten ... and there's eleven and nine.

54 Ishka: Mm.

55 Frances: Twelve and eight or have we already done [inaud] ten ways?

56 Tim: You're discounting those are you Frances because you've already got them? Is that ... you went on to eleven and nine, and twelve and eight and so on. OK. So ten ways?

57 Ishka: Yup.

58 Tim: And you were counting them on your fingers? Yeh?

[laugh from Ishka]

59 Tim: OK. Um ... would it be fair Ishka to say that Frances did most of the work on, on that one, yeah? ... so I'm gonna give you another one and see if you can do most of the work on this, but make sure that Frances agrees with what you're doing. Erm, I'd like to know how many different ways there are of making the number, er, thirteen ... thirteen.

60 Ishka: Thirty? [emphasises the y]

61 Tim: Thirteen. [emphasises the n]

62 Frances: Thirteen?

63 Ishka: Yup ... thirteen. Erm, there's ten and three, twelve and ... one, eleven and two ... shall we write them down so we don't forget them.

64 Frances: So yes. What did you say?

65 Ishka: Well there's ...

66 Tim: You can start again if you want, right.

67 Ishka: Twelve and one.

68 Frances: Mm?

69 Ishka: Twelve and one, eleven and two, nine and four, um, ten and three ... no we've had that.

70 Frances: No we haven't.

71 Ishka: Yeh.

72 Frances: You forgot to say the three that's all.

73 Tim: [laughs] You added a three onto the ten you'd already written down! Yes, OK.

74 Frances: OK.

75 Ishka: And then, erm, eight and ... nine and five ... no not! [sighs] ... nine and ...

76 Frances: You already did nine. See, nine and four.

77 Ishka: Oh yeah, er ... oh yeah I was doing eight and what was it, um, eight and five.

78 Frances: OK. Next? [Frances assumes the "teacher/listener" role]

79 Ishka: Seven and six.

80 Frances: Seven and six, yes.

81 Ishka: Um ...

82 Frances: Er, is that all?

83 Ishka: I think so.

84 Tim: Can you think of any others Frances? No?

85 Frances: They would be the same.

86 Tim: So how many have we got?

87 Ishka: One, two, three, four, five, six.

88 Tim: OK, so for the number thirteen there are six ways.

89 Ishka: Yes.

90 Tim: Let's just remember what we got for the others. For ten there was ...

91 Frances: We got

92 F+Ishka: five.

93 Tim: For twenty?

94 Frances: Ten.

95 Tim: And for thirteen?

96 Ishka: Six.

97 Tim: OK. Now ... supposing I said, um ... this is the same question right? ... how many different ways are there of finding two numbers that add up to, but this time I'll say, um, thirty.

98 Frances: Thirty?

99 Tim: Thirty, three zero. OK.

100 Frances: Yup.

101 Tim: But instead of going through them as you did before and counting them up, I'd like you to make, er, if you like, a prediction as to how many you think there'll be.

102 Frances: OK.

103 Tim: Mm-hm.

104 Frances: Thirty.

105 Ishka: I think there'll be around ...

106 Frances: Fifteen?

107 Ishka: Yup.

108 Frances: Maybe?

109 Ishka: Fifteen because ...

110 Tim: Did you ...

111 Ishka: that's half and

112 Tim: Ah hah.

113 Ishka: Most of them are half or just about one away from ...

114 Frances: Six and ... thirteen.

115 Tim: That's, that's er, that's, that's ... your prediction's thirty? [misheard] ... and Ishka's is ...

116 Frances: I mean fifteen.

117 Tim: [laughs] OK. Prediction of fifteen. Er, so ... lets just go back to what Ishka was saying. She was saying that in most cases it's about half.

118 Ishka: Well, yes, 'cause ten was five.

119 Tim: Right.

120 Frances: and /twenty was ten/.

121 Ishka: [almost simulltaneous] /and then ... twenty was /ten.

122 Tim: Yes.

123 Frances: and /thirteen was about six/.

124 Ishka: /but , erm thirteen /was six.

125 Tim: OK.

126 Ishka: Although that isn't exactly half.

127 Frances: But that would be six and a half if it was equalled.

128 Tim: OK, what about, erm - I'll choose an outrageous one now - what about if we did it for a hundred ... numbers that add up to a hundred. How many ways?

129 Frances: Fifty?

130 Ishka: About fifty yeah.

131 Tim: About fifty. Now are you saying about fifty, Ishka, because you're sort of playing safe or I mean do you really think it is fifty?

132 Ishka: Well maybe not exactly, but it's around fifty basically?

133 Tim: OK. And Frances do you think it's exactly fifty or around fifty?

134 Frances: Maybe around fifty.

135 Tim: OK. What about this prediction of fifteen for er thirty, was that around fifteen or exactly fifteen?

136 Ishka: It's fifteen or around.

137 Frances: Yes.

138 Ishka: 'cause we can't be exactly sure until we've tried it, but ...

139 Tim: OK. It might be an idea then to ... I mean, that isn't too big to just check them is it. Yeah? Would you like to have a go between you? Either write it

down or say them or whatever you want, OK?

140 Ishka: OK so ...

141 Frances: Who's going to say half? [writes "prediction 15"]

142 Ishka: Are we doing fifteen?

143 Tim: We're doing, you're doing numbers that make thirty.

144 F+Ishka: Thirty.

145 Ishka: OK so that's fifteen and fifteen. [F writes 15+15]

146 Frances: Fifteen and fifteen.

147 Ishka: Er,

148 Frances: Let's go down.

149 Ishka: OK. Er, fourteen and sixteen.

150 Frances: Yes [F continues the list]

151 Ishka: Thirteen and seventeen.

152 Frances: Yes.

153 Ishka: Then, er, fourteen and ...

154 Frances: You already did fourteen.

155 Ishka: Oh yeah. Um, twelve and [sighs; pause]

156 Tim: What will it be Frances?

157 Frances: Eighteen.

158 Tim: OK.

159 Ishka: So twelve and eighteen. Um ...

160 Frances: Eleven and? [the 'teacher' again]

161 Ishka: Eleven and nineteen.

162 Frances: Good. Ten?

163 Ishka: And twenty. And um? ...

164 Frances: Nine?

165 Ishka: And twentyone.

166 Frances: Eight?

167 Ishka: And twentytwo.

168 Frances: Seven?

169 Ishka: Twentythree.

170 Frances: Six?

171 Ishka: Twentyfour.

172 Frances: Seven, eight, er, five?

[Tim laughs]

173 Ishka: And twentyfive.

174 Frances: [sighs] Four?

175 Ishka: Twentysix.

176 Frances: Three?

177 Ishka: Twentyseven.

178 Frances: Two?

179 Ishka: Let's see ... and twentyeight.

180 Frances: One?

181 Ishka: And twentynine.

182 Frances: Good.

[They count how many pairs there are]

183 Ishka: One, two.

184 Frances: One, two, three, four, [and up to] fourteen, fifteen. So we were right.

185 Tim: So it's exactly fifteen, yup. Incidentally ... oh, just a tiny little thing Frances - when you do your sevens, you write them and then cross off a little bit at the beginning. Why do you do that?

186 Frances: Oh er, I kept doing it the Japanese way because I got used to doing it the Japanese way because I've lived in Japan.

187 Tim: Yeah, but I think, I mean I think that's alright, don't you Ishka?

188 Ishka: What like that?

189 Tim: That little bit on the seven.

190 Ishka: What you mean like this? [draws it]

191 Frances: Yes.

192 Tim: Yeah.

193 Ishka: Well I usually, I either do my sevens like that, like that ...

194 Tim: All different ways, yeah. I think it's not a problem Frances, that's all I'm saying. Yeah, yeah. Um, now, you know when you'd written them all down you counted, you started at the top and went one, two, three, four, five ... can you notice something about these numbers that would immediately tell you there's fifteen without writing them all down?

195 Frances: There's one to fifteen, so fifteen numbers?

196 Tim: OK. The left hand column starts with one and goes up to fifteen yup? OK.

197 Frances: Otherwise sixteen would be the same as si ... as fourteen and sixteen.

198 Tim: Right. If you did a similar listing for, um, let's say ... because we asked about, we thought about a hundred didn't we? - and how many ways there were of doing that. If you did a list like this and you started at ... what would be the two equal numbers at the top?

199 Ishka: Fifty and fifty.

200 Tim: Fifty add fifty, and then it would go fortynine ...

201 Frances: [sighs] fortynine fiftyone, fortyeight fiftytwo.

202 Tim: And so on?

203 I+Frances: Yup.

204 Tim: And the last one would be?

205 Frances: One.

206 Tim: Right.

207 Frances: /One and ninetynine/.

208 Ishka /A hundred, no, ninetynine/.

209 Tim: And how many numbers would there be in those, or how many ways would you have done it?

210 F+Ishka: Fifty.

211 Tim: OK? So you, now you know it's fifty, it's not about fifty, OK?

212 Ishka: So the even numbers are exactly half.

213 Tim: That's really good Ishka.

214 Ishka: And the odd numbers are ...

215 Frances: Are exactly half.

216 Ishka: Are sort of, are one off ... that.

217 Tim: Er, say what you ... I know what you mean. So like with thirteen it's ...

218 Ishka: Like with thirteen it was six, in half ...

219 Tim: Mm hm.

220 Ishka: So it couldn't be exactly because it was odd.

221 Tim: Right.

222 Ishka: So they went to twelve, which was the one underneath and split that in half.

223 Frances: er , so do you mean that twelve ... is six and six, so do you mean like if it's twentynine or thirteen, if the, if it's an odd number, they have ... they have more odd numbers than the e ... even numbers? Is that what you mean?

224 Ishka: I'm not really sure what you mean. [laughs]

225 Tim: Well suppose take, take your ...

226 Frances: Suppose, if there's um thirteen, so there's .... one, two, three ... [counts under her breath, whilst making 13 pencil tally marks] ... thirteen ... if it's like this ...

227 Tim: Mm hm.

228 Frances: This is, these are ... odd [indicating alternate tallies 1st, 3rd etc] and these [others] are even, so you mean that there're more odds than there're evens. Mm? If it's, if thirteen is ... [sighs] ... If the number is an odd number do you mean that there's, there would normally be more odd numbers than evens?

229 Ishka: I don't know.

230 Tim: Do you understand, Frances is saying that if you take all the numbers from one up to thirteen, yeah? And if you look at all those numbers one, two, three, four, five, up to thirteen, there's more odd numbers in that list than even numbers. How many more Frances?

231 Frances: Erm, normally one more.

232 Tim: Normally one more. [laughs] Can you think of an example where it's not exactly one more?

233 Frances: Hmm ...

234 Tim: No, nor can I! But if I said with a number, erm, twentythree, OK? - how many ways of making it from adding two numbers?

235 Frances: [whispers] twentythree, twentythree. Errm ...

236 Ishka: Eleven?

237 Tim: Eleven. How did you get that Ishka?

238 Ishka: Well, um, from when we done thirteen if we take one down which is, twenty...[swallows]

239 Frances: Two.

240 Ishka: Two, then split that in half.

241 Tim: Right.

242 Frances: Twentytwo divided by two is eleven.

243 Ishka: Yeah that's it.

244 Tim: Right, OK, I just wonder whether it's possible by writing down all the ways of getting, say, thirteen like this ... yuh? ... whether you could then see why it was half of the number before. I don't know, do you think it might be possible?

245 Frances: Yes.

246 Tim: Do you want to try it Frances?

247 Ishka: So we start at half of thirteen.

248 Frances: Half of thirteen would be ...

249 Ishka: Six.

250 Frances: Six and seven. Six point five ...[laughs]

251 Tim: You can't have six and a half and six and a half, so what will you start with?

252 Frances: Six and seven.

253 Tim: Six and seven. Mm hm. Want to write it down? [F begins list] And the next one?

254 Frances: Five and eight.

255 Tim: We can already see why it must be now, can't we.

256 Frances: [mumble] Four plus nine, three plus ten, two plus eleven, one plus twelve.

257 Tim: OK, so it's six yeah?

258 Ishka: Yeah.

259 Tim: And it's not se ... why didn't we write down seven and six

260 Frances: Because if you wrote seven and six it's just this turned the other way round.

261 Tim: So there's no point on putting it down. Yeh?

262 F/Ishka: No.

263 Frances: This is just ... [draws arrows transposing 6, 7]

264 Tim: OK, OK. Thank you very much we've finished then.

END OF SESSION

TRANSCRIPT IRG5: ALLAN May 1995

Allan is an 'average' Year 9 (13-14 years) pupil in an Essex comprehensive 11-16 school. Judith is in her first year as a teacher. She was a member of the IRG, and donated this transcript.

1 Judith: OK, Now. Here is the investigation, so you can read it. Draw a three by three dot grid. Start anywhere you like. Draw a continuous line that goes to every dot. Yeah?

2 Allan: Yeah.

3 Judith: This one is nine units long - one, two, three, four, fiv,e six, seven, eight, nine. 'K, no diagonals. Find the shortest possible line on a three by three grid. Investigate for larger squares. Right, so how are you going to start off?

4 Allan: Em ... [pause]

5 Judith: What are you thinking? [pause]

6 Allan: Well first I'm going to find a suitable starting position on the grid.

7 Judith: OK. [pause] So you're starting in the corner, are you?

8 Allan: Yeah, I'm gonna start in the top left-hand corner.

9 Judith: So tell me why you thought that you would start in the top left-hand corner.

10 Allan: Well, it seems a reasonable spot because um, from the top left-hand corner it'd be quite easy to go right round, like nine dots and try-, and get to a, and get to all the corners, quite easily.

11 Judith: OK, Do you think you're going to be able to do it in less than ...that? ...nine?

12 Allan: Maybe, yeah.

13 Judith: Maybe

14 Allan: Maybe

15 Judith: OK, so you're not ...

16 Allan: Not positive, but I am ...

17 Judith: OK [pause] top right [pause] Are you thinking about where to go next?

18 Allan: Yeah

19 Judith: So how many is it?

20 Allan: eight

21 Judith: OK, do you think that's the shortest from starting in the corner? Try another one.

22 Allan: Yeah I'm (indistinguishable mumble) [pause]

23 Judith: ' many?

24 Allan: Still eight again so probably the most is eight.

25 Judith: You mean the least?

26 Allan: Yeah, the least, sorry.

27 Judith: That's allright.

28 Allan: Yeah.

29 Judith: OK, so do you think starting not in the corner could get you eight as well?

30 Allan: Um, possibly, yeah. [pause]

31 {start speaking together}

32 Judith: Pardon?

33 Allan: ' got a problem with one of the corners, (mumble) dots.

34 Judith: Why was it you said you'd got a problem with one of the corners?

35 Allan: I missed out one of the corners.

36 Judith: Oh right. [pause]

37 Judith: ' many was that?

38 Allan: That'd be nine.

39 Judith: Yep. So what are you going to try now ... or what do you think?

40 Allan: So it don't work from the middle at all, really, because it's uh, because you have to go in and out again.

41 Judith: D'you think? Are you sure?

42 Allan: Because what you have to do, you have to go, from the middle you have to go all the way round and you go into one and come back out again, but if you do it from, like we 'ave did from exactly, exactly in the middle

43 Judith: Try that then. [pause]

44 Allan: Yeah, made eight as well from the middle

45 Judith: OK. So what do you think? [pause] So if I wasn't talking to you and you were just doing the investigation yourself, what would you be thinking?

46 Allan: I'd be thinking that the most possible um way uh of getting, of getting, getting the least is that it only started from the middle going right round, going right round or going, going from one corner but you can't do it in the middle, middle between two corners

47 Judith: OK

48 Allan: and also the least is eight

49 Judith: Allright then, so what're you going to do now?

50 Allan: I'll try a, um, four by four grid.

51 Judith: Right. Can you make any predictions before you start?

52 Allan: Um

53 Judith: Or ...

54 Allan: The maximum will probably be, er, the least'll probably be 'bout fifteen.

55 Judith: Right. And anything else you're going to notice?

56 Allan: Uh..no, not really

57 Judith: OK [pause, Allan tries]

58 Judith: So what was that?

59 Allan: fifteen

60 Judith: Can you do it in less?

61 Allan: I'll give it a go. [pause] I done it in fourteen.

62 Judith: Count again

63 Allan: fifteen

64 Judith: So why did you ... so do you think your prediction was right?

65 Allan: Yeah

66 Judith: Are you sure?

67 Allan: Yeah.

68 Judith: So why did you predict fifteen?

69 Allan: Uh because I thought there might be a pattern between ... if there was um a certain amount of, um ... if it's three by three say

70 Judith: Uh-hum

71 Allan: if you ti-, three times three is actually nine

72 Judith: uh-hum

73 Allan: But as, if you went round all the dots, it would only come to about, if you did it once it would come to on-, uh less than nine, 'n' you got, uh, because, because there's o-, there's only ... 'cause you only have, y- ... you can miss out a line exactly, 'cause y- you can miss out a gap, c- 'cause you um, y'd 'ave to go all the way round the whole dots

74 Judith: OK ... So why did that make you say fifteen

75 Allan: because uh, f- for the same reason, 'cause if you um w- tried to go round the whole all the dots you'd get sixteen but if you just did it once all the way round the dots but missing out gaps you'd still come to uh, you just minus one basically and just ...

76 Judith: So what would happen in some other squares?

77 Allan: Probably if you minus one from the s-, if you square the number you'd probably find that if it was actually, if you minus one from that you'd probably find that that would be the answer to the ...

78 Judith: OK

79 Allan: to how many dots there are, to how many times you 'ave to go round the dots

80 Judith: OK. Now do you want to try one or are you certain of that?

81 Allan: No, I'll give it a go with a five by five

82 Judith: Yup, Good [pause]

83 Allan: The twenty-five one it, 'ts diff- ... er, it's tweny-six.

84 Judith: Count, again

85 Allan: It's twenty-five

86 Judith: Is that what you expected?

87 Allan: Yes, it was.

88 Judith: So if you were doing an investigation what would you write down for me?

89 Allan: I'd write ... that the pattern is ... if you ti-, times both, if you square the side, the side, and um, you minus one, you'd be th', the amount of dots you-, if you went round the dots it'd be the same answer.

90 Judith: Very good. OK, now what do-, it says 'Investigate for larger squares', what do you think you could investigate now for me? Now you've done squares, Allan, you've given me a rule for squares, and, you could, um, extend this investigation to look at something else, anything you want. Any ideas?

91 Allan: [pause]

92 Judith: Do you want me to give you some? [pause]

93 Allan: No I haven't got any ideas.

94 Judith: Rectangles?

95 Allan: Yeah

96 Judith: Can you make predictions about rectangles first before you draw them?

97 Allan: Could um ... four by six?

98 Judith: Yup

99 Allan: [pause] twenty-three for a four by six rectangle

100 Judith: Why

101 Allan: 'Cause I'm basing it on the squ-, same pattern as the square and not , not just work [pause]

102 Judith: So what happened?

103 Allan: It came to twenty-seven.

104 Judith: And was that what you expected?

105 Allan: No

106 Judith: So what's happening in the rectangle that it's not what you expected? And you've counted it twice, did you say?

107 Allan: Yeah, it's twenty-seven.

108 Judith: OK, and what size is your square?

109 Allan: It's four by six

110 Judith: four by six what?

111 Allan: Um ... [pause] centimetre dots.

112 Judith: What do you mean?

113 Allan: Um ... [pause] There's six lines across by four lines.

114 Judith: OK, now I've noticed that you've made a mistake somewhere there, but I want you to see wh-, see if you can notice it, or if you want to go on to other ... rectangles and see what's happened ... OK now it didn't come to what you've predicted. Is that 'cause you've counted wrong, or you've drawn your square, or rectangle rather, wrong, or what?

115 Allan: [pause] I think it may be because I've predicted wrong.

116 Judith: OK, try another one, try maybe a more simple one, what size is this one going to be?

117 Allan: It's going to be a four by five.

118 Judith: Allright. So there's four dots down one side, and

119 Allan: Five dots on the top side

120 Judith: OK. [pause] ,So according to your original prediction what would it have been?

121 Allan: Would've been twenty-four

122 Judith: Right, and what's your prediction now?

123 Allan: Um [pause]

124 Judith: Hold on, four by five? Twenty-four?

125 Allan: No, no, the-

126 Judith: With the original prediction it would've been ...

127 Allan: Yeah, um, twenty-four.

128 Judith: Why?

129 Allan: We-, well I did it on the same basis as the um, um square, the squ-, if you um, times both sides, the amount of dots, you um minus by one it might be the same amount.

130 Judith: OK, so if you times the sides you get what?

131 Allan: For which one?

132 Judith: This one, this wee one

133 Allan: It'd be twenty.

134 Judith: Right, and if you minus one it'd be

135 Allan: It'd be nineteen.

136 Judith: Right, so you, your original prediction for this would've been nineteen, but

you told me that you thought you'd predicted wrong, correcting

137 Allan: Yeah

138 Judith: Right, so what do you think this is going to be now?

139 Allan: [pause] It may be twenty-three.

140 J; Right, try it. [pause] OK, count them.

141 Allan: It's nineteen.

142 Judith: 'Tis nineteen, right I'll tell you that you didn't, well, the size of your rectangle was wrong, so look carefully at the size of your rectangle on the first one again. How many dots down is it?

143 Allan: Four

144 Judith: And how many dots across?

145 Allan: seven

146 Judith: Right, so see what you did?

147 Allan: Yeah

148 Judith: Was your prediction right?

149 Allan: Yeah, it would've been, yeah

150 Judith: Right, so can you tell me, make any general predictions about rectangles, or not?

151 Allan: Yeah, it's, it's the same as the square, you um, times it by, ti-, times the both sides together, and minus one

152 Judith: OK, now, here's some triangle dotty paper, hold on, are you sure that would work or not, first? Do you want to try another one?

153 Allan: Yeah

154 Judith: Yeah, you'd better.

155 Allan: I'll do a six by five.

156 Judith: OK, which should be what?

157 Allan: twenty-nine

158 Judith: Yup [pause]

159 Allan: Come to twenty-nine.

160 Judith: So ... what would you write down if you were doing an investigation?

161 Allan: I would write, the pattern is, the same as a square, which is, you times the two si-, t-times the length by the width, as if area, and you mi- you minus one um, you just minus one from the total

162 Judith: OK and that'll give you the shortest possible line

163 Allan: Yeah.

164 Judith: Sure?

165 Allan: Yeah.

166 Judith: OK. Let's talk about triangles then. So I've got some triangle dotty, and this time you can go in that direction or that direction or that direction, so if you start at that dot you can go to any of the dots near it, you can't go diagonally like that

167 Allan: Yup

168 Judith: OK, now I want you to look at triangles, so you pick a size of triangle and then tell me the shortest line to join all the dots in it. Where's your first triangle gonna be, and how big's it gonna be?

169 Allan: I'm gonna have a ... a three by three triangle

170 Judith: Right. Draw it for me. First of all draw round your triangle so I can see where it is. OK, now what's the shortest line to join all the dots?

171 Allan: It'd be five

172 Judith: Why?

173 Allan: [pause] Um ... that one'd be ... that'd be six, 'cause you have to miss out one, one line, miss out one line, you'd have to do one line at one corner

174 Judith: Right, so how many does that make?

175 Allan: And what it is, it is two on, two on each side, and three more in fact, and two on the other two sides

176 Judith: So how many lines are there?

177 Allan: There'd be five lines

178 Judith: OK, and how many dots would give you five lines?

179 Allan: six.

180 Judith: Right. Do you want to try another triangle? What size?

181 Allan: Yup, um, I want to try five by five

182	Judith:	Any predictions?
183	Allan:	Y--eah [pause] Actually I don't.
184	Judith:	Don't know enough yet.
185	Allan:	No.
186	Judith:	OK [pause]
187	Allan:	Um, it came to fourteen.
188	Judith:	And how many dots were there?
189	Allan:	Um ... There were fifteen dots.
190	Judith:	OK, what are you going to try now?
191	Allan:	Um, I'm gonna give a, I'll try six by six
192	Judith:	OK. Now, any predictions?
193	Allan:	[pause] Uh, seve-, I think that there'd be seventeen lines
194	Judith:	seven teen?
195	Allan:	Yeah
196	Judith:	OK, Why?
197	Allan:	Because I think there's a pattern that if you add all the sides together, that and minus one, it'd come to, it'd i-, um, that'd be the amount of lines there are to go round it.
198	Judith:	OK, did it work for two or three dots?
199	Allan:	[pause] Yeah, it did
200	Judith:	So tell me exactly what you do with three?
201	Allan:	Um, you add two, y'add, you do two add two add two, and then you just minus one which comes to five but it was originally six
202	Judith:	Right, and what do you do with the five?
203	Allan:	You do five add five add five, and minus one, which comes to fourteen.
204	*Judith:*	*Right, the only problem* with that is that when you had a triangle that had three along one side you added two and two and two, and when *you had* five along one side you added five and five and five. Do you see what I'm saying, no?
205	Allan:	Yeah.

206 Judith: Sort of?

207 Allan: Yeah

208 Judith: Right, well let's try your six anyway and see if that was right what you said, I mean it could still be right. [pause] So how many was it?

209 Allan: It was twenty.

210 Judith: And you would've predicted seventeen?

211 Allan: Yeah.

212 Judith: So how many dots were there?

213 Allan: Um ... twenty-one

214 Judith: Right, what do you notice? Anything?

215 Allan: Yeah, if you minus one, it, it's actually in dots, that um ... when you start getting dots inside the triangle, it, you it's the amount of dots, but before that it, it's only, it's the only amount of lines.

216 Judith: R--ight. Explain that more.

217 Allan: Um, When you've got triangles that are like one by one by one, you 'aven't got any dots in the middle at all, then you wouldn't have to do the dots 'cause it'd work both ways anyway really,

218 Judith: Um-hum

219 Allan: But if you did the ones that had the dots inside them, like um when you come to about four by , four by five er, four by four by four, they start er, er, and they have dots in the middle, you start having to do it by dots.

220 Judith: Right. And how do you do it by dots?

221 Allan: You count the, count the amount of dots that go round the whole, i- that, um, in the triangle and outside the triangle as well, and then you just minus one.

222 Judith: OK, so you've done three and five and six dots along the outside. Do four for me. Can you tell me what you think it's going to be?

223 Allan: Um, yeah. eleven.

224 Judith: Right [pause] OK. Right, I mean it is quite difficult, do you want to draw me a table and see if you can see any patterns, you've done three, four, five and six now. Do that now.What did you get just there? So what did, what are the headings for your table going to be?

225 Allan: Um, Size of triangle

226 Judith: What do you mean by size of triangle?

227 Allan: How many dots there are, uh, dots around the edges..

228 Judith: OK. So the first column is..

229 Allan: Size of triangle.

230 Judith: And then ... OK, well fill in the different sizes of triangle first. [pause] Now if you did it three, four, five, six it might be easier to see patterns, do you not think? Do you do tables when you do investigations, I can't remember what you do. OK, anyway, what's the next one going to be? Well we need one for the shortest line, don't we?

231 Allan: Yeah.

232 Judith: OK so what ... [pause]

233 Allan: Well that tells what the shortest line is in the, going round in the triangle

234 Judith: OK

235 Allan: Basically [pause]

236 Judith: um-hum What the shortest line is, is going to be? [pause] OK, four was nine, five was..

237 Allan: Fourteen.

238 Judith: six was..

239 Allan: six was twenty

240 Judith: OK, well from that can you see any patterns to do with those numbers five, nine, fourteen, twenty that could, give you a way of predicting what seven would be? So what sort of things could you look for? Tell me what you're doing out loud, in your head.

241 Allan: Um, I'm seeing if, that if you minus fourteen from twe- uh, that if you added fou-, four to five it'd, to make nine, and you added, a four to nine to make fourteen,

242 Judith: Uh-hum

243 Allan: four to fourteen's twenty, twenty-six, it don't work

244 Judith: Does it work?

245 Allan: No [pause]

246 Judith: So what, what else can you try? [pause] What do you think? Could you times by anything?

247 Allan: [pause] No, you can't.

248 Judith: OK, what else could you try? [pause] Confused?

249 Allan: Yeah.

250 Judith: So if I gave you those numbers: five, nine fourteen, twenty and asked you what came next .... [pause] What do you think? What are you trying?

251 Allan: [pause] Erm, I think I know the pattern, what it is, is if you add four to five, you get nine, if you add five to fourteen, you add five to nine you get fourteen, if you add six to fourteen you get twenty, if you add seven to twenty you get twenty-seven.

252 Judith: Right, and what would eight be?

253 Allan: threefive

254 Judith: OK, so, what oth-, so, there was a long time you were quiet, I want to know what sort of things you were thinking when you were doing that, what sort of things did you try?

255 Allan: I was uh ..., well actually I was working out whether the pattern I was trying was actually working or not, I was just going through it to find out.

256 Judith: So that was the first thing you tried, then?

257 Allan: Yeah.

258 Judith: OK, good. So do you want to see if that works for seven then?

259 Allan: Yeah.

260 Judith: So you think eight, or for eight, which one are you going to do seven or eight, 'cause you've told me what both of them you think should be.

261 Allan: Yeah seven.

262 Judith: OK. [pause]

263 Allan: Yeah it came to twenty-seven.

264 Judith: Very good, Well done!

END OF SESSION

TRANSCRIPT NT4: JONATHAN 13th March 1995

Jonathan is an undergraduate Mathematics/Education student, who has chosen a third year option in the Theory of Numbers. The paper is assessed by a 3-hour paper and two 25-hour 'projects'. Jonathan has chosen to work on the problem of finding the number of integer solutions of $x^2+y^2=n$ mod p. At a previous supervision he had discussed the cases $n=0, 1$ with me, including a proof for the case $n=0$. Since then he has generated some data for $n>1$.

1 Jonathan: Well, I had a bit of a bash this time with the theory.

2 Tim: Right .. we're talking about x squared plus y squared equals n ...

3 Jonathan: Yes

4 Tim: Yes, yes ... um, can you ... did we discuss equals zero last time?

5 Jonathan: It ... came up, yes.

6 Tim: Right ... you'll forgive me, but I've discussed the same/

7 Jonathan: /yeah (yes)/

8 Tim: question with one or two people.

9 Jonathan: Yes, and ... that .... that I'm quite ha .... well, fairly happy with the argument I can put for that one.

10 Tim: [rising pitch] Alright.

11 Jonathan: That's, that's, that's the happiest argument that I've got [laughs].

12 Tim: Right, right. And, um, tell me what the theorem is, in that case then, about x squared plus y squared equals zero.

13 Jonathan: x squared plus y squared equals zero, there, there's either one pair of ... one pair of solutions that does it, nought, nought,

14 Tim: Mm, hmm

15 Jonathan: or, there's ... quite a lot of them

16 Tim: yeah (yes)

17 Jonathan: depending on whether you're congruent to one or three mod p.

18 Tim: [non-committal, rising] Right ...

19 Jonathan: And the 'quite a lot of them' turns out to be two p minus one. [pause] Where p's a prime.

20 Tim: yeah (yes).

21 Jonathan: Um, I'm just trying to find the page with my argument on it. Um ..

22 Tim: S.. so .. but it's two p minus one ... but you mentioned whether or not p was congruent to one or three mod four [*NB he didn't*] that, in fact it's two p minus one in either case, is it?

23 Jonathan: Oh, no, sorry, it's, um, if it's congruent to one mod four,

24 Tim: Right

25 Jonathan: then there are two p minus one /solutions/

26 Tim: /Oh, sorry,/ yes, but if it's congruent to three then there's just one.

27 Jonathan: Just the one pair.

28 Tim: OK, OK. And, I mean, can I, I think, I just want to askdoes it hinge on the fact that in one case minus one is a quadratic residue and in the other case it isn't?

29 Jonathan: [pause] Um ... well, yes [coughs] ... sort of. Um, I mean /it's, yes there's one/

30 Tim: /[laughs] Would/ you like to rehearse the argument with me, or ...

31 Jonathan: Well [coughs], yeah (yes), I'll come back to that bit about the quadratic residue bit. Um, but for where it's equal to one mod four ...

32 Tim: right

33 Jonathan: Or, in both cases we know that nought squared and nought squared is going to be there ..

34 Tim: Right

35 Jonathan: So that gives us one guaranteed solution for/

36 Tim: /right/

37 Jonathan: any [inaudible]/

38 Tim: right

39 Jonathan: Um, and then, if p is congruent to one [pause] we've then got ... p minus, p minus one is a quadratic residue of that.

40 Tim: Indeed; or minus one.

41 Jonathan: Whichever way. So that gives us, um, one squared and p minus one squared, so we've got another pair of solutions.

42 Tim: Umm ... you don't quite mean that do you? I mean, you mean, what you've

got, one squared and whatever gives you p minus one ...

43 Jonathan: Oh, yes, so ..

44 Tim: ... when it's squared. Right, /OK/

45 Jonathan: /OK/

46 Tim: Yeah?

47 Jonathan: And, then there's this pairing thing ..

48 Tim: Yeah?

49 Jonathan: Which ...that's the bit I can't, I'm not ... able to explain. I can't, I'm not, I can't say why they pair off, like that. Um, but then(?) we've got, um, p minus one over two pairs [*number of quadratic residues mod p*] [inaudible]

50 Tim: Oh, p minus one over two squares.

51 Jonathan: Yes. And so, so you get [long pause] yes, sorry, yes that's it. And they add up to give p each time, these two .. these pairs of squares ..

52 Tim: yes

53 Jonathan: So you've got p there, nought.

54 Tim: [pause] Um, [hesitant] that's an absolutely fine ... um, I mean, let's think, we're talking about when p is congruent with one mod four here, aren't we?

55 Jonathan: Um, yes.

56 Tim: Yeah, so, for example, thirteen.

57 Jonathan: Right.

58 Tim: So you're saying .. um, um .. I'm trying to think of something that isn't [*-13*] ... well, no, let's have something fairly straightforward. If you do two squared you get four, yeah?

59 Jonathan: Right.

60 Tim: And you're saying that in fact, um, thirteen minus four, or something congruent to that, minus four, mod thirteen, is always - in fact it's nine ...

61 Jonathan: Yes.

62 Tim: ...three squared - is always there. So you're saying, in that case, there always happen to be pairs that add to ...

63 Jonathan: Yes.

64 Tim: OK. I mean, can you take it any further than there. I mean you're absolutely right. How can you take it any further than "there always happens to be"?

65 Jonathan: No, I'm stumbling on this, but this is, this is the the bit that, It's sort of an assumption I have to make [exhaled laugh] to go through this, and I ..

66 Tim: OK

67 Jonathan: .... I can't ...

68 Tim: OK

69 Jonathan: I can't ... and I know ... or, I don't know ... from looking at the ones that are congruent to three, mod four ..

70 Tim: Yes

71 Jonathan: ... that there's not a constant adding up, for the pairs, so I can see that the two ..

72 Tim: Right.

73 Jonathan: ... they really do separate, but I can't explain, why they separate.

74 Tim: OK. Well, I'd like to take you a bit further down that road, because I think you'll be quite pleased when you see it. OK?

75 Jonathan: Right.

76 Tim: I'm just wondering whether to talk about thirteen, or something that's less obvious. You know, 'cos (because) [laughs]....

77 Jonathan: Oh dear [laughs], is thirteen obvious! [laughs]

78 Tim: No, no, no, I mean, um, an argument can be more forceful when you can't just - other than the numerical calculations - say "Well, obviously". Yes?

79 Jonathan: Yes.

80 Tim: So, um, suppose ...

81 Jonathan: Yes?

82 Tim: So suppose ... [pause] Right,OK, here's a good ... well, this isn't quite .... suppose we took seventeen.

83 Jonathan: Yes, OK.

84 Tim: And the first thing to note is, you should know that minus one is a quadratic residue, and that's a particularly easy one because four squared is sixteen, which is minus one.

85 Jonathan: Oh, yes.

86 Tim: So just bear in mind if you will that four squared is minus one, yes?

87 Jonathan: Right.

88 Tim: Now pick - you pick, anything from nought to sixteen.

89 Jonathan: Ten.

90 Tim: Ten. So ten tens are a hundred, OK?

91 Jonathan: Right.

92 Tim: Which mod seventeen, er, happens to be, well it's fifteen actually.

93 Jonathan: That's about right.

94 Tim: OK? So you want to find something whose square is two, yes? Or to put it another way, something whose square is minus fifteen.

95 Jonathan: Right.

96 Tim: Yeah. Now, note that you know that ten squared is fifteen.

97 Jonathan: Um, hm.

98 Tim: And you know that four squared is minus one [Tim writes $10^2=15$, $4^2=-1$]. Can you put those two together to get something whose square is minus fifteen?

99 Jonathan: [long pause] Forty mod seventeen?

100 Tim: Forty?

101 Jonathan: Yes?

102 Tim: And how did you get that?

103 Jonathan: Well, I'm guessing here but if it was anything else I'd go: well, I want fifteen [*he means minus fifteen*] and I know what minus one is, so minus one times fifteen is minus fifteen. So I've taken the four and the ten and multiplied them.

104 Tim: But can you see that's not ... that's more than a guess isn't it? That, you know, if x squared is a and y squared is b, then xy all squared /is ab/

105 Jonathan: /Oh yes/, if the [inaudible] would say it was a guess.

106 Tim: [laughs] OK

107 Jonathan: There were just two [inaudible]

108 Tim: OK, so now just to confirm it; forty, what is forty mod seventeen?

109 Jonathan: Um, six.

110 Tim: OK. Six squared?

111 Jonathan: Which is two.

112 Tim: OK.

113 Jonathan: Right.

114 Tim: [pause]. Now, can you see that in principle you can do that with absolutely ... I mean, you took ten, but you could do that with anything that you took.

115 Jonathan: Yeah.

116 Tim: And it's because, minus one's a quadratic residue.

117 Jonathan: Oh ... right [chuckles].

118 Tim: Yeah, yeah.

119 Jonathan: That's slightly, yes, slightly reassuring now.

120 Tim: Well, it takes us beyond this kind of level of "there always happens to be one". I mean, that's why there always has to be one.

121 Jonathan: Right, OK.

122 Tim: Now, um, what I'm not going to do for you is, when p is congruent to three mod four, you know that you've got nought and nought, you know that minus one is not a quadratic residue. Right? Why is it that you can never find two numbers whose squares add to zero?

123 Jonathan: OK.

124 Tim: The fact that minus one isn't a quadratic residue ... I mean, you need a bit more argument to show why you can never find solutions other than zero zero in that case.

125 Jonathan: Right.

126 Tim: [pause] That's quite nice, I think. It kind of wraps up, at least most of that, in quite a tidy way.

127 Jonathan: Yes, it takes out some of the arm waving from the argument ...[Jonathan and Tim laugh] which is no bad thing.

128 Tim: Well, arm waving's alright, as long as it, as long as you're saying "I could do this if you asked, if I needed to".

129 Jonathan: "It can easily be shown that" ... [*laughs, an in-joke?*]

130 Tim: Ye-es.

131 Jonathan: I recognise that one!

132 Tim: Yes, yes. [pause]

133 Jonathan: Right.

134 Tim: Yes, I mean "It can be shown that" ... is OK if you mean what you say, you know, rather than, um ... I mean, that's beyond a conjecture.

135 Jonathan: Yes, I felt fairly certain that there was a ... reasonable way of showing that those were .. seeing that regular pattern, there had to be a good way of explaining it.

136 Tim: OK, OK. [pause] So which then brings us, I guess, back to x squared plus y squared equals one, does it?

137 Jonathan: Uh, yes. Yes. That's a very intriguing way of ... well, actually no, I went on to do something else first.

138 Tim: Ah?

139 Jonathan: Of x squared plus y squared equals n mod p, but n not being zero and having a particular value.

140 Tim: OK.

141 Jonathan: Um, having found out how many solutions there were for that ...

142 Tim: 'cos (because) originally you took it to be anything other than zero...

143 Jonathan: Yes.

144 Tim: Which actually is not without interest, if I may say so.

145 Jonathan: No, I think I did go on to show, um ...

146 Tim: I mean, now you know how many solutions there are for zero, you can say precisely how many there are for not-zero [laughs]

147 Jonathan: Yes, that's basically what I'd done! [laughs]

148 Tim: OK.

149 Jonathan: p squared and taken away how many other bits there are ...

150 Tim: Excellent.

151 Jonathan: ... so I've done that.

152 Tim: Yes, that wraps up quite nicely too.

153 Jonathan: Yes, that was very satisfying, actually.

154 Tim: Yes. I think it's just a nice coincidence that you've two p minus one solutions, and when you subtract that from p squared, you get a perfect square ...

155 Jonathan: Oh, yes ...

156 Tim: p squared minus two p plus one is a, an algebraic square.

157 Jonathan: Ooh! So when I looked at those I was finding, just from doing examples, x squared plus y [*omits squared*] equals two, or whatever, that there were the same number of solutions as there were to x squared plus y squared equals one, congruent to one.

158 Tim: Yes, OK, OK.

159 Jonathan: So for that, um, I said, well, all you're doing is adding solution pairs, or like multiplying solution pairs again. So if you've got ...a solution pair for one ...

160 Tim: Yes, yes.

161 Jonathan: And you multiply it by two, it then becomes a solution pair for x squared plus y squared is congruent to two.

162 Tim: [pause] Do you mean that?

163 Jonathan: I think I do.

164 Tim: I mean, I was getting quite excited until you said "two", on the end [laughs]. Um, can I take an example, right?

165 Jonathan: I've got one somewhere, I think.

166 Tim: OK.

167 Jonathan: Just scrambling here, a little bit ... ah yes, I've got one in two and in ... where are we? [pause]. No I think I may have said something that doesn't, isn't quite right. Um, I've got two and six, a solution pair to x squared plus y squared is congruent to one ... mod thirteen. Then, is two times that a solution to x squared plus y squared is congruent to two mod thirteen? So I've taken two lots of two squareds and two lots of six squareds, like that ... which then is congruent to ....

168 Tim: Alright, two squared plus two squared plus six squared plus six squared ...

169 Jonathan: Right, by doing it twice.

170 Tim: Ok, but that's not the same as something squared plus something squared

is two. [pause]. I mean, you haven't from that got a solution to x squared plus y squared equals two.

171 Jonathan: Um [pause] No [pause] Oh drat! [laughs]

172 Tim: Well, don't despair 'cause (because) ...[pause] you've got two squared plus six squared is one, right?

173 Jonathan: Um-hm.

174 Tim: Now what I was thinking of, suppose you doubled two and you doubled six, so take four squared plus twelve squared . Right then, what ... what's that equal to? [pause] Mod thirteen?

175 Jonathan: [pause, then laughs]. Let me think. [long pause]

176 Tim: [quietly] Would you like a calculator then?

177 Jonathan: [firmly, joking] No! [both laugh]. Er, four.

178 Tim: OK. And, er, tell me when you're ready [Jonathan is making notes]. I don't want to interrupt ...

[*gap here as the tape is turned over*]

179 Tim: ... so you've multiplied the two and the six by two, and therefore the one becomes four. OK.

180 Jonathan: Yes.

181 Tim: But you could in principle do that with every solution to x squared plus y squared equals one, to get a solution to x squared plus y squared equals four. Yes?

182 Jonathan: Yes, OK.

183 Tim: And nine, the same ...

184 Jonathan: And nine. Yeah.

185 Tim: Yeah? Or indeed any square on the right hand side. And by square, that means quadratic residue. Right? I mean, let's think of something that isn't an obvious square but is a quadratic residue. What are we doing, mod thirteen? Well alright, three.

186 Jonathan: Right.

187 Tim: Four fours are sixteen so three's a quadratic residue. So you could use that method to get, um, all the solutions of x squared plus y squared is three, right?

188 Jonathan: Right.

189 Tim: So if you've got x squared plus y squared equals n, where n is a quadratic residue ...

190 Jonathan: OK.

191 Tim: ... then again you know there are as many solutions to that as there are to x squared plus y squared equals one [pause whilst Jonathan makes notes]

192 Jonathan: Right.

193 Tim: Now, let me just put two things to you. First of all, how do you know that there aren't any more, right? So for every solution to x squared plus y squared equals one you get one to x squared plus y squared equals ... n, where n is a quadratic residue, but how do you know there aren't any more?

194 Jonathan: Right.

195 Tim: And I suppose the way to do that would be, to, sort of, work the argument backwards in some way.

196 Jonathan: Right.

197 Tim: That whenever you've got a solution to x squared plus y squared equals n, where n is a quadratic residue, that must arise from a solution to x squared plus y squared equals one.

198 Jonathan: Right.

199 Tim: So that's, that's number one thing for you to think about, right, to work on. And the second thing for you to think about and work on, is when n is not a quadratic residue ...

200 Jonathan: Right.

201 Tim: .. that argument doesn't apply, at least not in the form that I gave it ...

202 Jonathan: Right.

203 Tim: Can I tell you that I don't know the answer to this [both laugh]. I mean simply because I've not allowed myself to think about it ...

204 Jonathan: Yes, very restrained of you. [both laugh]. As soon as I'm out of that door you'll be going [rest inaudible beneath Tim's laughter]

205 Tim: Nice of you to suggest it. Um ...

206 Jonathan: Anything to put off sorting through that pile of papers! [gestures to Tim's desk]

207 Tim: [laughs] Dead right! Um, right ... yes, if n is not a quadratic residue that argument does not apply in the form that I gave it. And yet, the conclusion holds, about there being the same number of solutions [pause]. And that's a, quite a curious one, really. So I'll leave that with you, OK?

208 Jonathan: Right. Uh ... OK.

209 Tim: You know the other thing that, um, you haven't proved - but in a way I don't feel to desperate about it because there's quite a lot around here for you to write up - is ... is why there are ... one more or one less than p solutions to x squared plus y squared equals one, in every case.

210 Jonathan: Ah well, ah, that's, I'm coming on to that bit ...

211 Tim: Ah, right! Sorry ..

212 Jonathan: I had to backtrack to get to that.

213 Tim: Oh, right. OK, OK.

214 Jonathan: This is, this is ... Once I'd found out that there were the same number of solutions for, um, for all the other values ..

215 Tim: Right, right ..

216 Jonathan: Um ... I'm can then get back to some serious arm waving here and ... and go back to my sort of proof of why there are x ... there are p plus one or p minus one solutions.

217 Tim: Right.

218 Jonathan: And, basically, um, you say how many ... you take your mod, number ...

219 Tim: Right ..

220 Jonathan: And you work out how many possible pairs you can come up with ....

221 Tim: [hushed] Right ...

222 Jonathan: And ... whatever that was, p squared ...

223 Tim: OK.

224 Jonathan: We already know how many... solutions there are - for p congruent to one or congruent to three mod four - how many solutions there are for ... x squared plus y squared is congruent to zero [pause]

225 Tim: [hushed] Yeah ...

226 Jonathan: So we can get rid of those for starters. And then we know that all the solutions that are left are divided up evenly between each of the other

numbers ...

227 Tim: Ohh, that's very nice. [Jonathan laughs]. Oh, well done!

228 Jonathan: Which was a little arm waving bit there ...

229 Tim: That's not arm waving at all.

230 Jonathan: I thought it was a bit contrived, but ...

231 Tim: Does it come out with the right answer?

232 Jonathan: It comes out with exactly the right answer, yes.

233 Tim: Oh, well done, well done. Yes ... [laughs]

234 Jonathan: I didn't know if that was the way you were thinking of ...

235 Tim: That's very nice indeed.

236 Jonathan: ... of attacking it, but um ...

237 Tim: Um it's ... well, I can tell you that I've discussed this problem with Bob Hall, and that was his approach to it ...

238 Jonathan: Oh, right [laughs]

239 Tim: Mine wasn't ... that but, it's very nice, no, that's very nice. OK.

240 Jonathan: Came as a blinding inspirational thought.

241 Tim: Right. Well, I would suggest you rush away and write all this down.

242 Jonathan: [laughs] Well, I wrote it all down yesterday, that ... that particular bit ..

243 Tim: Right.

244 Jonathan: ... about how to, er, do it, so it's just the, um, it's the pairing up of the quadratics when it's congruent and ... that tidying up of the, er, x squared [inaudible].

245 Tim: OK. I think this is looking extremely good, right, so I think it's quite important that you now get down ... as coherent an account as you can, of what you've done ... before you go on to do anything else.

246 Jonathan: Right. It's taken a second battering now, it's, er [...]

247 Tim: Rush away and write it up.

248 Jonathan: Yes, before I forget it!

249 Tim: [laughs] OK. Well that was a good way to start the week.

END OF SESSION